INTEXT QUESTIONS 1.2

HOW WAS YOUR UNITS AND DIMENSIONS CHAPTER?

Notes

1. Experiments with a simple pendulum show that its time period depends on its length (l) and the acceleration due to gravity (g). Use dimensional analysis to obtain the dependence of the time period on l and g .

2. Consider a particle moving in a circular orbit of radius r with velocity υ and acceleration a towards the centre of the orbit. Using dimensional analysis, show that $a \propto \upsilon^2/r$.

3. You are given an equation: $m\upsilon = Ft$, where m is mass, υ is speed, F is force and t is time. Check the equation for dimensional correctness.

1.4 VECTORS AND SCALARS

1.4.1 Scalar and Vector Quantities

In physics we classify physical quantities in two categories. In one case, we need only to state their magnitude with proper units and that gives their complete description. Take, for example, mass. If we say that the mass of a ball is 50 g, we do not have to add anything to the description of mass. Similarly, the statement that the density of water is 1000 kg m^{-3} is a complete description of density. Such quantities are called scalars. **A scalar quantity has only magnitude; no direction.**

On the other hand, there are quantities which require both magnitude and direction for their complete description. A simple example is velocity. The statement that the velocity of a train is 100 km h^{-1} does not make much sense unless we also tell the direction in which the train is moving. Force is another such quantity. We must specify not only the magnitude of the force but also the direction in which the force is applied. Such quantities are called vectors. **A vector quantity has both magnitude and direction.**

Some examples of vector quantities which you come across in mechanics are: displacement (Fig. 1.3), acceleration, momentum, angular momentum and torque etc.

What is about energy? Is it a scalar or a vector?

To get the answer, think if there is a direction associated with energy. If not, it is a scalar.

1.4.2 Representation of Vectors

A vector is represented by a line with an arrow indicating its direction. Take vector AB in Fig. 1.4. The length of the line represents its magnitude on some scale. The arrow indicates its direction. Vector CD is a vector in the same direction but its magnitude is smaller. Vector EF is a vector whose magnitude is the same as that of vector CD, but its direction is different. In any vector, the initial point, (point A in **AB),** is called the **tail** of the vector and the final point, (point B in **AB)** with the arrow mark is called its **tip** (or **head**).

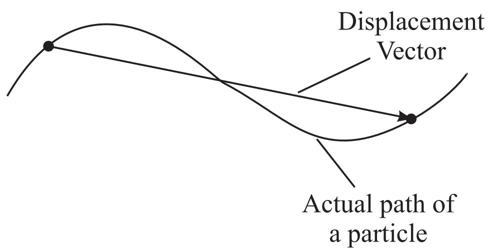


Fig. 1.3 : Displacement vector

A vector is written with an arrow over the letter representing the vector, for example, $\vec{A}$. The magnitude of vector $\vec{A}$ is simply A or $|\vec{A}|$. In print, a vector is indicated by a bold letter as **A**.

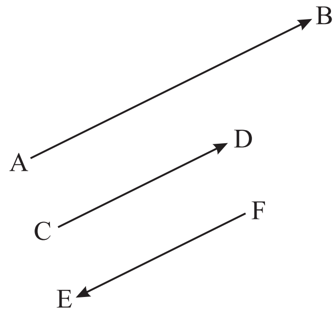


Fig. 1.4 : Directions and magnitudes of vectors

Two vectors are said to be equal if their magnitudes are equal and they point in the same direction. This means that all vectors which are parallel to each other have the same magnitude and point in the same direction are equal. Three vectors **A, B** and **C** shown in Fig. 1.5 are equal. We say **A = B = C**. But **D** is not equal to **A**.

A vector (here **D**) which has the same magnitude as A but has opposite direction, is called **negative** of **A**, or **–A**. Thus, **D** = –**A**.

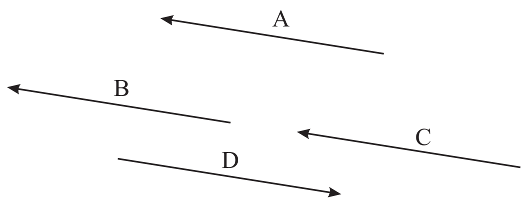


Fig. 1.5 : Three vectors are equal but fourth vector D is not equal.

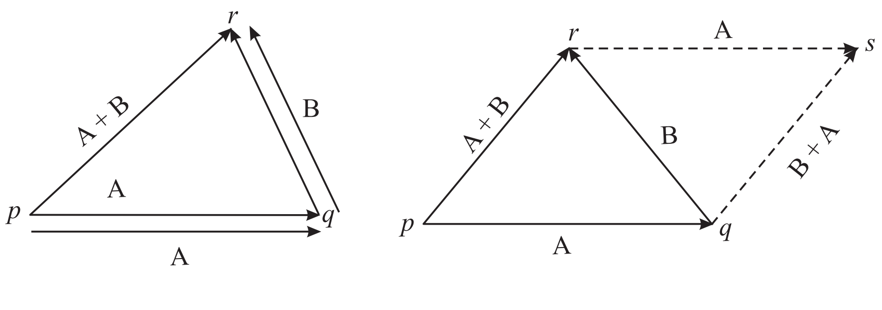
For respresenting a physical vector quantitatively, we have to invariably choose a proportionality scale. For instance, the vector displacement between Delhi and Agra, which is about 300 km, is represented by choosing a scale 100 km = 1 cm, say. Similarly, we can represent a force of 30 N by a vector of length 3cm by choosing a scale 10N = 1cm.

From the above we can say that if we translate a vector parallel to itself, it remains unchanged. This important result is used in addition of vectors. Let us sec how.

1.4.3 Addition of Vectors

You should remember that only **vectors of the same kind can be added**. For example, two forces or two velocities can be added. But a force and a velocity cannot be added.

Suppose we wish to add vectors **A** and **B.** First redraw vector **A** [Fig. 1.6 (a)]. For this draw a line (say *pq*) parallel to vector **A.** The length of the line i.e. *pq* should be equal to the magnitude of the vector. Next draw vector **B** such that its tail coincides with the tip of vector **A**. For this, draw a line *qr* from the tip of **A** (i.e., from the point *q*) parallel to the direction of vector **B.** The sum of two vectors then is the vector from the tail of **A** to the tip of **B,** i.e. the resultant will be represented in magnitude and direction by line *pr.* You can now easily prove that **vector addition is commutative. That is, A + B = B + A,** as shown in Fig. 1.6 (b). In Fig. 1.6(b) we observe that *pqr is* a triangle and its two sides *pq* and *qr* respectively represent the vectors **A** and **B** in magnitude and direction, and the third side *pr*, of the triangle represents the resultant vector with its direction from *p* to *r*. This gives us a rule for finding the resultant of two vectors :



(a) (b)

Fig. 1.6 : Addition of vectors A and B

If two vectors are represented in magnitude and direction by the two sides of a triangle taken in order, the resultant is represented by the third side of the triangle taken in the opposite order. This is called triangle law of vectors.

The sum of two or more vectors is called the **resultant** vector. In Fig. 1.6(b), **pr** is the resultant of **A** and **B**. What will be the resultant of three forces acting along the three sides of a triangle in the same order? If you think that it is zero, you are right.

Let us now learn to calculate resultant of more than two vectors.

The resultant of more than two vectors, say **A**, **B** and **C**, can be found in the same manner as the sum of two vectors. First we obtain the sum of **A** and **B**, and then add the resultant of the two vectors, (**A** + **B**), to **C**. Alternatively, you could add **B** and **C**, and then add **A** to (**B** + **C**) (Fig. 1.7). In both cases you get the same vector. Thus, vector addition is associative. That is, **A + (B + C) = (A + B) + C.**

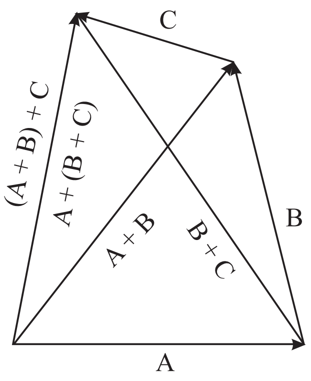


Fig. 1.7 : Addition of three vectors in two different orders

If you add more than three vectors, you will discover that **the resultant vector is the vector from the tail of the first vector to the tip of the last vector.**

Many a time, the point of application of vectors is the same. In such situations, it is more convenient to use parallelogram law of vector addition. Let us now learn about it.

1.4.4 Parallelogram Law of Vector Addition

Let **A** and **B** be the two vectors and let θ be the angle between them as shown in Fig. 1.8. To calculate the vector sum, we complete the parallelogram. Here side PQ represents vector **A**, side PS represents **B** and the diagonal PR represents the resultant vector **R**. Can you recognize that the diagonal PR is the sum vector **A** + **B**? It is called the **resultant** of vectors **A** and **B**. The resultant makes an angle α with the direction of vector **A**. Remember that vectors **PQ** and **SR** are equal to **A**, and vectors **PS** and **QR** are equal, to **B**. To get the magnitude of the resultant vector **R**, drop a perpendicular RT as shown. Then in terms of magnitudes

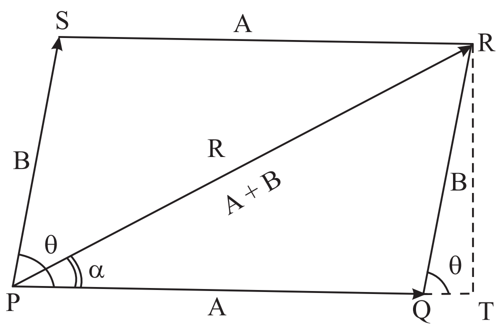


Fig. 1.8: Parallelogram law of addition of vectors

$$
\begin{aligned}
(PR)^2 &= (PT)^2 + (RT)^2 \\
&= (PQ + QT)^2 + (RT)^2 \\
&= (PQ)^2 + (QT)^2 + 2PQ.QT + (RT)^2 \\
&= (PQ)^2 + [(QT)^2 + (RT)^2] + 2PQ.QT \qquad (1.1) \\
&= (PQ)^2 + (QR)^2 + 2PQ.QT \\
&= (PQ)^2 + (QR)^2 + 2PQ.QR\ (QT / QR) \\
R^2 &= A^2 + B^2 + 2AB.\cos\theta
\end{aligned}
$$

Therefore, the magnitude of **R** is

$$|R| = \sqrt{A^2 + B^2 + 2AB.\cos\theta} \qquad (1.2)$$

For the direction of the vector **R**, we observe that

$$\tan\alpha = \frac{RT}{PT} = \frac{RT}{PQ + QT} = \frac{B\sin\theta}{A + B\cos\theta} \qquad (1.3)$$

So, the direction of the resultant can be expressed in terms of the angle it makes with base vector.

Special Cases

Now, let us consider as to what would be the resultant of two vectors when they are parallel?

To find answer to this question, note that the angle between the two parallel vectors is zero and the resultant is equal to the sum of their magnitudes and in the direction of these vectors.

Suppose that two vectors are perpendicular to each other. What would be the magnitude of the resultant? In this case, $\theta = 90^\circ$ and $\cos\theta = 0$.

Suppose further that their magnitudes are equal. What would be the direction of the resultant?

Notice that $\tan\alpha = B/A = 1$. So what is α?

Also note that when $\theta = \pi$, the vectors become anti-parallel. In this case $\alpha = 0$. The resultant vector will be along **A** or **B**, depending upon which of these vectors has larger magnitude.

Example 1.4: A cart is being pulled by Ahmed north-ward with a force of magnitude 70 N. Hamid is pulling the same cart in the south-west direction with a force of magnitude 50 N. Calculate the magnitude and direction of the resulting force on the cart.

Solution :

Here, magnitude of first force, say, A = 70 N.

The magnitude of the second force, say, B = 50 N.

Angle θ between the two forces = 135 degrees.

So, the magnitude of the resultant is given by Eqn. (1.2) :

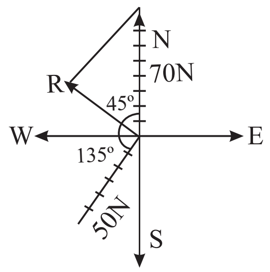


Fig. 1.9: Resultant of forces inclined at an angle

$$R = \sqrt{(70)^2 + (50)^2 + 2\times70\times50\times\cos(135)}$$

$$= \sqrt{4900 + 2500 - 7000\times\sin45}$$

$$= 49.5 \text{ N}$$

The magnitude of **R** = 49.5 N.

The direction is given by Eqn. (1.3):

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta} = \frac{50\times\sin(135)}{70 + 50\cos(135)} = \frac{50\times\cos 45}{70 - 50\sin 45} = 1.00$$

Therefore, α = 45.0° (from the tables). Thus **R** makes an angle of 45° with 70 N force. That is, **R** is in North-west direction as shown in Fig. 1.9.

1.4.5 Subtraction of Vectors

How do we subtract one vector from another? If you recall that the difference of two vectors, **A – B**, is actually equal to **A + (–B)**, then you can adopt the same method as for addition of two vectors. It is explained in Fig. 1.10. Draw vector **–B** from the tip of **A**. Join the tail of **A** with the tip of **–B**. The resulting vector is the difference **(A – B)**.

You may now like to check your progress.

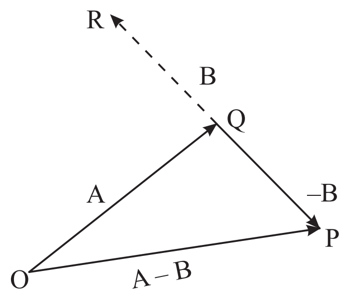
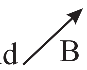


Fig. 1.10 : Subtraction of vector B from vector A

INTEXT QUESTIONS 1.3

Given vectors 

and 

1. Make diagrams to show how to find the following vectors:

   (a) **B – A**, (b) **A + 2B**, (c) **A – 2B** and (d) **B – 2A**.

2. Two vectors **A** and **B** of magnitudes 10 units and 12 units are anti-parallel. Determine **A + B** and **A – B**.
3. Two vectors **A** and **B** of magnitudes A = 30 units and B = 60 units respectively are inclined to each other at angle of 60 degrees. Find the resultant vector.

1.5 MULTIPLICATION OF VECTORS

1.5.1 Multiplication of a Vector by a Scalar

If we multiply a vector **A** by a scalar k, the product is a vector whose magnitude is the absolute value of k times the magnitude of **A**. This means that the magnitude of the resultant vector is $k\,|\mathbf{A}|$. The direction of the new vector remains unchanged if k is positive. If k is negative, the direction of the new vector is opposite to its original direction. For example, vector 3**A** is thrice the magnitude of vector **A**, and it is in the same direction as **A**. But vector –3**A** is in a direction opposite to vector **A**, although its magnitude is thrice that of vector **A**.

1.5.2 Scalar Product of Vectors

The **scalar product** of two vectors **A** and **B** is written as **A.B** and is equal to AB cosθ, where θ is the angle between the vectors. If you look carefully at Fig. 1.11, you would notice that B cosθ is the projection of vector **B** along vector **A**. Therefore, the scalar product of **A** and **B** is the product of magnitude of **A** with the length of the projection of **B** along **A**. Another thing to note is that even if we take the angle between the two vectors as 360 – θ, it does not matter because the cosine of both angles is the same. Since a dot between the two vectors indicates the scalar product, it is also called the **dot product**. **Remember that the scalar product of two vectors is a scalar quantity.**

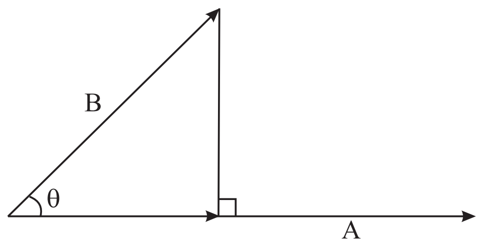


Fig. 1.11: Projection of B on A

A familiar example of the scalar product is the work done when a force **F** acts on a body moving at an angle to the direction of the force. If **d** is the displacement of the body and θ is the angle between **F** and **d**, then the work done by the force is $Fd\cos\theta$.

Since dot product is a scalar, it is commutative: **A.B** = **B.A** = ABcosθ. It is also distributive: **A.(B + C) = A.B + A.C**.

1.5.3 Vector Product of Vectors

Suppose we have two vectors **A** and **B** inclined at an angle θ. We can draw a plane which contains these two vectors. Let that plane be called Ω ((Fig. 1.12 a)

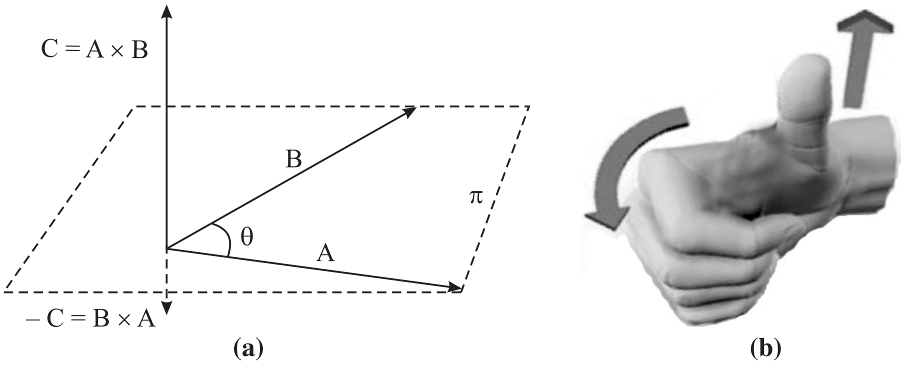


Fig.1.12 (a) : Vector product of Vectors; (b) Direction of the product vector C =A × B is given by the right hand rule. If the right hand is held so that the curling fingers point from A to B through the smaller angle between the two, then the thumb strectched at right angles to fingers will point in the direction of C.

which is perpendicular to the plane of paper here. Then the vector product of these vectors, written as **A** × **B**, is a vector, say **C**, whose magnitude is AB sinθ and whose direction is perpendicular to the plane Ω. The direction of the vector **C** can be found by **right-hand rule** (Fig. 1.12 b). Imagine the fingers of your right hand curling from **A** to **B** along the smaller angle between them. Then the direction of the thumb gives the direction of the product vector **C**. If you follow this rule, you can easily see that direction of vector **B** × **A** is opposite to that of the vector **A** × **B**. This means that **the vector product is not commutative**. Since a cross is inserted between the two vectors to indicate their vector product, the vector product is also called the cross product.

A familiar example of vector product is the angular momentum possessed by a rotating body.

To check your progress, try the following questions.

INTEXT QUESTIONS 1.4

1. Suppose vector **A** is parallel to vector **B**. What is their vector product? What will be the vector product if **B** is anti-parallel to **A**?
2. Suppose we have a vector **A** and a vector **C** = $\frac{1}{2}$ **B.** How is the direction of vector **A** × **B** related to the direction of vector **A** × **C**.
3. Suppose vectors **A** and **B** are rotated in the plane which contains them. What happens to the direction of vector **C** = **A** × **B**.
4. Suppose you were free to rotate vectors **A** and **B** through arbitrary amounts keeping them confined to the same plane. Can you make vector **C** = **A** × **B** to point in exactly opposite direction?

5. If vector **A** is along the x-axis and vector **B** is along the y-axis, what is the direction of vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$? What happens to **C** if **A** is along the y-axis and **B** is along the x-axis?
6. **A** and **B** are two mutually perpendicular vectors. Calculate (a) $\mathbf{A} \cdot \mathbf{B}$ and (b) $\mathbf{A} \times \mathbf{B}$.

1.6 RESOLUTION OF VECTORS

Resolution of vectors is converse of addition of vectors. Here we calculate components of a given vector along any set of coordinate axes. Suppose we have vector **A** as shown in Fig. 1.13 and we need to find its components along x and y-axes. Let these components be called A_x and A_y respectively. Simple trigonometry shows that

$$A_x = A\cos\theta \tag{1.4}$$

and

$$A_y = A\sin\theta, \tag{1.5}$$

where θ is the angle that **A** makes with the x - axis. What about the components of vector **A** along X and Y-axes (Fig. 1.13)? If the angle between the X-axis and **A** is φ, then

$$A_X = A\cos\phi$$

and

$$A_Y = A\sin\phi.$$

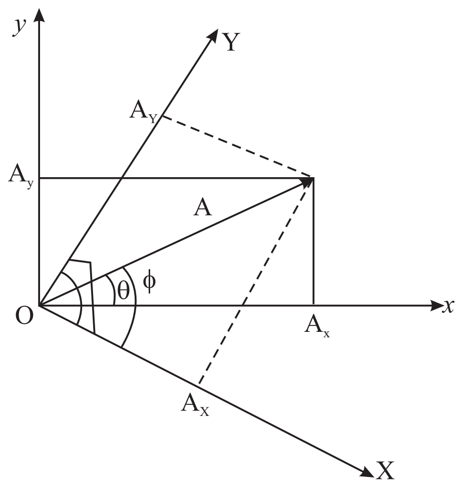


Fig. 1.13 : Resolution of vector A along two sets of coordinates (x, y) and (X, Y)

It must now be clear that the components of a vector are not fixed quantities; they depend on the particular set of axes along which components are required. Note also that the magnitude of vector **A** and its direction in terms of its components are given by

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{A_X^2 + A_Y^2} \tag{1.6}$$

and $\tan\theta = A_y / A_x$, $\tan\phi = A_Y / A_X$. (1.7)

So, if we are given the components of a vector, we can combine them as in these equations to get the vector.

1.7 UNIT VECTOR

At this stage we introduce the concept of a **unit vector**. As the name suggests, a unit vector has unitary magnitude and has a specified direction. It has no units and no dimensions. As an example, we can write vector **A** as A $\hat{\mathbf{n}}$ where a cap on **n** (i.e. $\hat{\mathbf{n}}$) denotes a unit vector in the direction of **A**. Notice that a unit vector has been introduced to take care of the direction of the vector; the magnitude has been taken care of by A. Of particular importance are the unit vectors along coordinate axes. Unit vector along x-axis is denoted by $\hat{\mathbf{i}}$, along y-axis by $\hat{\mathbf{j}}$ and along z-axis by $\hat{\mathbf{k}}$. Using this notation, vector **A**, whose components along x and y axes are respectively A_x and A_y, can be written as

$$\mathbf{A} = A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}\,. \qquad (1.8)$$

Another vector **B** can similarly be written as

$$\mathbf{B} = B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}}\,. \qquad (1.9)$$

The sum of these two vectors can now be written as

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\,\hat{\mathbf{i}} + (A_y + B_y)\,\hat{\mathbf{j}} \qquad (1.10)$$

By the rules of scalar product you can show that

$$\hat{\mathbf{i}}.\,\hat{\mathbf{i}}=1,\ \hat{\mathbf{j}}.\hat{\mathbf{j}}=1,\ \hat{\mathbf{k}}.\hat{\mathbf{k}}=1,\ \hat{\mathbf{i}}.\hat{\mathbf{j}}=0,\ \hat{\mathbf{i}}.\hat{\mathbf{k}}=0, \text{ and } \hat{\mathbf{j}}.\hat{\mathbf{k}}=0 \qquad (1.11)$$

The dot product between two vectors **A** and **B** can now be written as

$$\begin{aligned}\mathbf{A}\,.\,\mathbf{B} &= (A_x\,\hat{\mathbf{i}} + A_y\,\hat{\mathbf{j}}).\,(B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}}) \\ &= A_xB_x\,(\hat{\mathbf{i}}\,.\,\hat{\mathbf{i}}) + A_xB_y\,(\hat{\mathbf{i}}\,.\,\hat{\mathbf{j}}) + A_yB_x\,(\hat{\mathbf{j}}\,.\,\hat{\mathbf{i}}) + A_yB_y\,(\hat{\mathbf{j}}\,.\,\hat{\mathbf{j}}) \\ &= A_xB_x + A_yB_y, \qquad (1.12)\end{aligned}$$

Here, we have used the results contained in Eqn. (1.11).

Example 1.4: On a coordinate system (showing all the four quadrants) show the following vectors:

$$\mathbf{A} = 4\hat{\mathbf{i}} + 0\hat{\mathbf{j}},\ \mathbf{B} = 0\hat{\mathbf{i}} + 5\hat{\mathbf{j}},\ \mathbf{C} = 4\hat{\mathbf{i}} + 5\hat{\mathbf{j}},$$

$$\mathbf{D} = 6\hat{\mathbf{i}} - 4\hat{\mathbf{j}}.$$

Find their magnitudes and directions.

Notes

Solution : The vectors are given in component form. The factor multiplying $\hat{\mathbf{i}}$ is the x component and the factor multiplying $\hat{\mathbf{j}}$ is the y component. All the vectors are shown on the coordinate grid (Fig. 1.14).

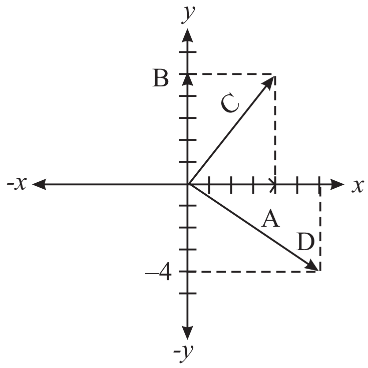


Fig. 1.14

The components of **A** are $\mathbf{A}_x = 4$, $\mathbf{A}_y = 0$. So, the magnitude of $\mathbf{A} = 4$. Its direction is $\tan^{-1}\left(\frac{A_y}{A_x}\right)$ in accordance with Eqn. (1.7). This quantity is zero, since $A_y = 0$. This makes it to be along the x-axis, as it is. Vector **B** has x-component = 0, so it lies along the y-axis and its magnitude is 5.

Let us consider vector **C**. Here, $\mathbf{C}_x = 4$ and $\mathbf{C}_y = 5$. Therefore, the magnitude of **C** is $C = \sqrt{4^2+5^2} = \sqrt{41}$. The angle that it makes with the x-axis is $\tan^{-1}(\mathbf{C}_y/\mathbf{C}_x)$ = 51.3 degrees. Similarly, the magnitude of **D** is $D = \sqrt{60}$. Its direction is $\tan^{-1}(\mathbf{D}_y/\mathbf{D}_x) = \tan^{-1}(0.666) = -33.7°$ (in the fourth quadrant).

Example 1.5: Calculate the product **C . D** for the vectors given in Example 1.4.

Solution : The dot product of **C** with **D** can be found using Eqn. (1.12):

$$\mathbf{C}\,.\,\mathbf{D} = C_xD_x + C_yD_y = 4\times6 + 5\times(-4) = 24 - 20 = 4.$$

The cross product of two vectors can also be written in terms of the unit vectors. For this we first need the cross product of unit vectors. For this remember that the angle between the unit vectors is a right angle. Consider, for example, $\hat{\mathbf{i}}\times\hat{\mathbf{j}}$. Sine of the angle between them is one. The magnitude of the product vector is also 1. Its direction is perpendicular to the xy - plane containing $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, which is the z-axis. By the right hand rule, we also find that this must be the positive z-axis. And what is the unit vector in the positive z - direction. The unit vector $\hat{\mathbf{k}}$. Therefore,

$$\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}. \quad (1.13)$$

Using similar arguments, we can show,

$$\hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}},\ \hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}},\ \hat{\mathbf{j}}\times\hat{\mathbf{i}} = -\hat{\mathbf{k}},\ \hat{\mathbf{k}}\times\hat{\mathbf{j}} = -\hat{\mathbf{i}},\ \hat{\mathbf{i}}\times\hat{\mathbf{k}} = -\hat{\mathbf{j}}, \quad (1.14)$$

and $$\hat{\mathbf{i}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}}\times\hat{\mathbf{k}} = 0. \quad (1.15)$$

Example 1.6: Calculate the cross product of vectors **C** and **D** given in Example (1.4).

Solution : We have

$$\mathbf{C}\times\mathbf{D} = (4\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}}) \times (6\,\hat{\mathbf{i}} - 4\,\hat{\mathbf{j}})$$

$$= 24\,(\hat{\mathbf{i}}\times\hat{\mathbf{i}}) - 16\,(\hat{\mathbf{i}}\times\hat{\mathbf{j}}) + 30\,(\hat{\mathbf{j}}\times\hat{\mathbf{i}}) - 20\,(\hat{\mathbf{j}}\times\hat{\mathbf{j}})$$

Notes

Using the results contained in Eqns. (1.13 – 1.15), we can write

$$\mathbf{C} \times \mathbf{D} = -16\ \hat{\mathbf{k}} - 30\ \hat{\mathbf{k}} = -46\ \hat{\mathbf{k}}$$

So, the cross product of **C** and **D** is a vector of magnitude 46 and in the negative *z* direction. Since **C** and **D** are in the *xy*-plane, it is obvious that the cross product must be perpendicular to this plane, that is, it must be in the *z*-direction.

INTEXT QUESTIONS 1.5

1. A vector **A** makes an angle of 60 degrees with the *x*-axis of the *xy*-system of coordinates. If its magnitude is 50 units, find its components in *x*, *y* directions. If another vector **B** of the same magnitude makes an angle of 30 degrees with the X-axis of the XY- system of coordinates. Find its components now. Are they same as before?

2. Two vectors **A** and **B** are given respectively as $3\ \hat{\mathbf{i}} - 4\hat{\mathbf{j}}$ and $-2\ \hat{\mathbf{i}} + 6\hat{\mathbf{j}}$. Sketch them on the coordinate grid. Find their magnitudes and the angles that they make with the *x*-axis (see Fig. 1.14).

3. Calculate the dot and cross product of the vectors given in the above question.

You now know that each term in an equation must have the same dimensions. Having learnt vectors, we must now add this: **For an equation to be correct, each term in it must have the same character: either all of them be vectors or all of them be scalars.**

WHAT YOU HAVE LEARNT

- The number of significant figures determines the accuracy of a measurement.
- Every physical quantity must be measured in some unit and also expressed in this unit. The SI system has been accepted and followed universally for scientific reporting.
- Base SI units for mass, length and time are respectively kg, m and s. In addition to base units, there are derived units.
- Every physical quantity has dimensions. Dimensional analysis is a useful tool for checking correctness of equations.
- In physics, we deal generally with two kinds of quantities, scalars and vectors. A scalar has only magnitude. A vector has both direction and magnitude.
- Vectors are added according to the parallelogram rule.
- The scalar product of two vectors is a scalar.

The vector product of two vectors is a vector perpendicular to the plane containing the two vectors.
Vectors can be resolved into components along a specified set of coordinates axes.

4

MOTION IN A PLANE

In the preceding two lessons you have studied the concepts related to motion in a straight line. Can you describe the motion of objects moving in a plane, i.e, in two dimensions, using the concepts discussed so far. To do so, we have to introduce certain new concepts. An interesting example of motion in two dimensions is the motion of a ball thrown at an angle to the horizontal. This motions is called a *projectile motion.*

In this lesson you will learn to answer questions like : What should be the position and speed of an aircraft so that food or medicine packets dropped from it reach the people affected by floods or an earthquake? How should an athlete throw a discuss or a javelin so that it covers the maximum horizontal distance? How should roads be designed so that cars taking a turn around a curve do not go off the road? What should be the speed of a satellite so that it moves in a circular orbit around the earth? And so on.

Such situations arise in projectile motion and circular motion. Generally, circular motion refers to motion in a horizontal circle. However, besides moving in a horizontal circle, the body may also move in a vertical circle. We will introduce the concepts of angular speed, centripetal acceleration, and centripetal force to explain this kind of motion.

OBJECTIVES

After studying this lesson, you should be able to :

- *explain projectile motion and circular motion and give their examples;*
- *explain the motion of a body in a vertical circle;*
- *derive expressions for the time of flight, range and maximum height of a projectile;*
- *derive the equation of the trajectory of a projectile;*
- *derive expressions for velocity and acceleration of a particle in circular motion; and*
- *define radial and tangential acceleration.*

4.1 PROJECTILE MOTION

The first breakthrough in the description of projectile motion was made by Galileo. He showed that the horizontal and vertical motions of a slow moving projectile are mutually independent. This can be understood by doing the following activity.

Take two cricket balls. Project one of them horizontally from the top of building. At the same time drop the other ball downward from the same height. What will you notice?

You will find that both the balls hit the ground at the same time. This shows that the downward acceleration of a projectile is the same as that of a freely falling body. Moreover, this takes place independent of its horizontal motion. Further, measurement of time and distance will show that the horizontal velocity continues unchanged and takes place independent of the vertical motion.

In other words, the two important properties of a projectile motion are :

(i) a constant horizontal velocity component

(ii) a constant vertically downward acceleration component.

The combination of these two motions results in the curved path of the projectile.

Refer to Fig. 4.1. Suppose a boy at A throws a ball with an initial horizontal speed. According to Newton's second law there will be no acceleration in the horizontal direction unless a horizontally directed force acts on the ball. Ignoring friction of air, the only force acting on the ball once it is free from the hand of the boy is the force of gravity.

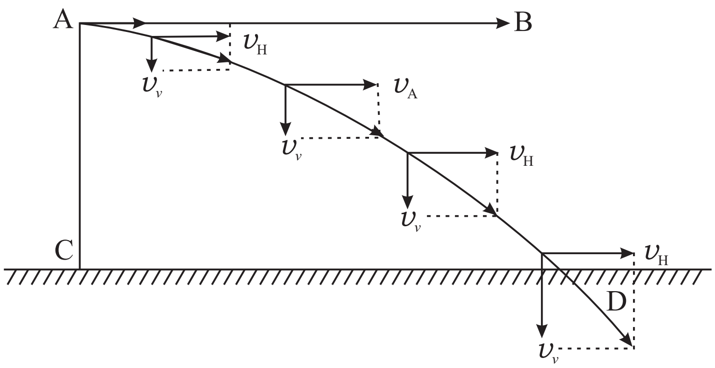


Fig. 4.1: Curved path of a projectile

Hence the horizontal speed v_H of the ball does not change. But as the ball moves with this speed to the right, it also falls under the action of gravity as shown by the vector's v_v representing the vertical component of the velocity. Note that $v = \sqrt{v_H^2 + v_v^2}$ and is tangential to the trajectory.

Having defined projectile motion, we would like to determine how high and how far does a projectile go and for how long does it remain in air. These factors are important if we want to launch a projectile to land at a certain target - for instance, a football in the goal, a cricket ball beyond the boundary and relief packets in the reach of people marooned by floods or other natural disasters.

4.1.1 Maximum Height, Time of Flight and Range of a Projectile

Let us analyse projectile motion to determine its maximum height, time of flight and range. In doing so, we will ignore effects such as wind or air resistance. We can characterise the initial velocity of an object in projectile motion by its vertical and horizontal components. Let us take the positive *x*-axis in the horizontal direction and the positive *y*-axis in the vertical direction (Fig. 4.2).

Let us assume that the initial position of the projectile is at the origin O at $t = 0$. As you know, the coordinates of the origin are $x = 0$, $y = 0$. Now suppose the projectile is launched with an initial velocity υ_0 at an angle θ_0, known as the ***angle of elevation,*** to the x-axis. Its components in the x and y directions are,

$$\upsilon_{ox} = \upsilon_o \cos\theta_0 \qquad (4.1\text{ a})$$

and

$$\upsilon_{oy} = \upsilon_o \sin\theta_0 \qquad (4.1\text{ b})$$

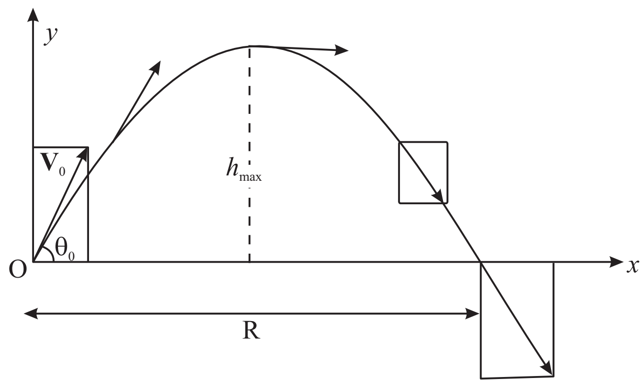


Fig 4.2 : Maximum height, time of flight and range of a projectile

Let a_x and a_y be the horizontal and vertical components, respectively, of the projectile's acceleration. Then

$$a_x = 0;\ a_y = -g = -9.8 \text{ m s}^{-2} \qquad (4.2)$$

The negative sign for a_y appears as the acceleration due to gravity is always in the negative y direction in the chosen coordinate system.

Notice that a_y is constant. Therefore, we can use Eqns. (2.6) and (2.9) to write expressions for the horizontal and vertical components of the projectile's velocity and position at time t. These are given by

Horizontal motion $\quad v_x = v_{ox}, \quad$ since $a_x = 0$ (4.3a)

$$x = v_{ox}t = v_0 \cos\theta_0 t \qquad (4.3b)$$

Vertical motion $\quad v_y = v_{oy} - g\,t = v_0 \sin\theta_0 - gt$ (4.3c)

$$y = v_{oy}t - ½g\,t^2 = v_0 \sin\theta_0 t - ½g\,t^2 \qquad (4.3d)$$

The vertical position and velocity components are also related through Eqn. (2.10) as

$$-g\,y = ½\,(v_y^2 - v_{oy}^2) \qquad (4.3e)$$

You will note that the horizontal motion, given by Eqns. (4.3a and b), is motion with constant velocity. And the vertical motion, given by Eqns. (4.3c and d), is motion with constant (downward) acceleration. The vector sum of the two respective components would give us the velocity and position of the projectile at any instant of time.

Now, let us make use of these equations to know the maximum height, time of flight and range of a projectile.

(a) Maximum height : As the projectile travels through air, it climbs upto some maximum height (h) and then begins to come down. ***At the instant when the projectile is at the maximum height, the vertical component of its velocity is zero.*** This is the instant when the projectile stops to move upward and does not yet begin to move downward. Thus, putting $v_y = 0$ in Eqns. (4.3c and e), we get

$$0 = v_{oy} - g\,t,$$

Thus the time taken to rise taken to the maximum height is given by

$$t = \frac{v_{oy}}{g} = \frac{v_0 \sin\theta_0}{g} \qquad (4.4)$$

At the maximum height h attained by the projectile, the vertical velocity is zero. Therefore, applying $v^2 - u^2 = 2$ a s $= 2\,g\,h$, we get the expression for maximum height:

$$h = \frac{v_0^2 \sin^2\theta_0}{2g} \qquad (\text{as } v = 0 \text{ and } u = v_0 \sin\theta) \qquad (4.5)$$

Note that in our calculation we have ignored the effects of air resistance. This is a good approximation for a projectile with a fairly low velocity.

Using Eqn.(4.4) we can also determine the total time for which the projectile is in the air. This is termed as the ***time of flight***.

(b) Time of flight : ***The time of flight of a projectile is the time interval between the instant of its launch and the instant when it hits the ground.*** The time t given by Eq.(4.4) is the time for half the flight of the ball. Therefore, the total time of flight is given by

$$T = 2t = \frac{2\,\upsilon_0 \sin\theta_0}{g} \tag{4.6}$$

Finally we calculate the distance travelled horizontally by the projectile. This is also called its ***range.***

(c) Range : The range R of a projectile is calculated simply by multiplying its time of flight and horizontal velocity. Thus using Eqns. (4.3b) and (4.4), we get

$$\begin{aligned} R &= (\upsilon_{ox})\ (2\,t) \\ &= (\upsilon_0 \cos\theta_0)\frac{(2\,\upsilon_0 \sin\theta_0)}{g} \\ &= \upsilon_0^2\,\frac{(2\sin\theta_0\,\cos\theta_0)}{g} \end{aligned}$$

Since $2\sin\theta\cos\theta = \sin 2\theta$, the range R is given by

$$R = \frac{\upsilon_0^2 \sin 2\theta_0}{g} \tag{4.7}$$

From Eqn. (4.7) you can see that the range of a projectile depends on

- its initial speed υ_0, and
- its direction given by θ_0.

Now can you determine the angle at which a disc, a hammer or a javelin should be thrown so that it covers maximum distance horizontally? In other words, let us find out the angle for which the range would be maximum?

Clearly, R will be maximum for any given speed when $\sin 2\theta_0 = 1$ or $2\theta_0 = 90^0$.

Thus, for R *to be maximum at a given speed* υ_0, θ_0 should be equal to 45^0.

Let us determine these quantities for a particular case.

Example 4.1 : In the centennial (on the occasion of its centenary) Olympics held at Atlanta in 1996, the gold medallist hammer thrower threw the hammer to a distance of 19.6m. Assuming this to be the maximum range, calculate the initial speed with which the hammer was thrown. What was the maximum height of the hammer? How long did it remain in the air? Ignore the height of the thrower's hand above the ground.

Solution : Since we can ignore the height of the thrower's hand above the ground, the launch point and the point of impact can be taken to be at the same height. We take the origin of the coordinate axes at the launch point. Since the distance covered by the hammer is the range, it is equal to the hammer's range for θ_0 = 45^0. Thus we have from Eqn.(4.7):

$$R = \frac{\upsilon_0^2}{g}$$

or $$\upsilon_0 = \sqrt{Rg}$$

It is given that R = 19.6 m. Putting g = 9.8 ms^{-2} we get

$$\upsilon_0 = \sqrt{(19.6\text{m})\times(9.8\text{ ms}^{-2})} = 9.8\sqrt{2}\text{ ms}^{-1} = 14.01\text{ms}^{-1}$$

The maximum height and time of flight are given by Eqns. (4.5) and (4.6), respectively. Putting the value of υ_0 and sin θ_0 in Eqns. (4.5) and (4.6), we get

$$\text{Maximum height, } h = \frac{\left(9.8\sqrt{2}\right)^2 \text{m}^2\text{s}^{-2} \times \left(\frac{1}{2}\right)^2}{2\times 9.8\text{ms}^{-2}} = 4.9\text{ m}$$

$$\text{Time of flight, } T = \frac{2\times\left(9.8\sqrt{2}\right)\text{m s}^{-1}}{9.8\text{ m s}^{-2}} \times \sqrt{\frac{1}{2}} = 2\text{ s}$$

Now that you have studied some concepts related to projectile motion and their applications, you may like to check your understanding. Solve the following problems.

INTEXT QUESTIONS 4.1

1. Identify examples of projectile motion from among the following situations:
   (a) An archer shoots an arrow at a target
   (b) Rocks are ejected from an exploding volcano
   (c) A truck moves on a mountainous road
   (d) A bomb is released from a bomber plane.
   [**Hint :** Remember that at the time of release the bomb shares the horizontal motion of the plane.]
   (e) A boat sails in a river.
2. Three balls thrown at different angles reach the same maximum height (Fig. 4.3):
   (a) Are the vertical components of the initial velocity the same for all the balls? If not, which one has the least vertical component?

(b) Will they all have the same time of flight?

(c) Which one has the greatest horizontal velocity component?

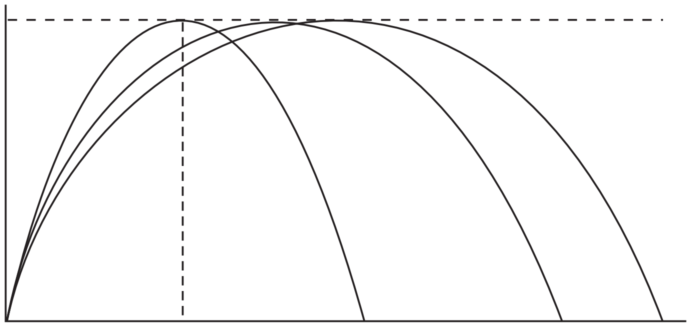


Fig. 4.3 : Trajectories of a projectile

3. An athelete set the record for the long jump with a jump of 8.90 m. Assume his initial speed on take off to be 9.5 ms^{-1}. How close did he come to the maximum possible range in the absence of air resistance?

   Take g = 9.78 ms^{-2}.

4.2 THE TRAJECTORY OF A PROJECTILE

The path followed by a projectile is called its trajectory. Can you recognise the shapes of the trajectories of projectiles shown in Fig. 4.1, 4.2 and 4.3.

Although we have discussed quite a few things about projectile motion, we have still not answered the question: What is the path or trajectory of a projectile? So let us determine the equation for the trajectory of a projectile.

It is easy to determine the equation for the path or trajectory of a projectile. You just have to eliminate *t* from Eqns. (4.3b) and (4.3d) for *x* and *y*. Substituting the value of *t* from Eqn. (4.3b) in Eqn.(4.3d) we get

$$y = \upsilon_{oy}\frac{x}{\upsilon_{ox}} - \frac{1}{2}\frac{g\,x^2}{\upsilon_{ox}^2} \left(\text{as } t = \frac{x}{\upsilon_{ox}}\right) \qquad (4.8\text{ a})$$

Using Eqns. (4.1 a and b), Eqn (4.8a) becomes

$$y = (\tan\theta_0)\,x - \frac{g}{2(\upsilon_0\cos\theta_0)^2}x^2 \qquad (4.8\text{ b})$$

as $\upsilon_{oy} = \upsilon_0 \sin\theta$ and $\upsilon_{ox} = \upsilon_0\cos\theta$.

Eqn. (4.8) is of the form $y = a\,x + b\,x^2$, which is the equation of a **parabola.** Thus, if air resistance is negligible, ***the path of any projectile launched at an angle to the horizontal is a parabola or a portion of a parabola.*** In Fig 4.3 you can see some trajectories of a projectile at different angles of elevation.

Eqns. (4.5) to (4.7) are often handy for solving problems of projectile motion. For example, these equations are used to calculate the launch speed and the angle of elevation required to hit a target at a known range. However, these equations do not give us complete description of projectile motion, if distance covered are very large. To get a complete description, we must include the rotation of the earth also. This is beyond the scope of this course.

Now, let us summarise the important equations describing projectile motion launched from a point (x_0, y_0) with a velocity υ_0 at an angle of elevation, θ_0.

Equations of Projectile Motion:

$$a_x = 0 \qquad\qquad a_y = -g \qquad (4.9\ a)$$

$$\upsilon_x = \upsilon_0 \cos\theta_0 \qquad\qquad \upsilon_y = \upsilon_0 \sin\theta - g\,t \qquad (4.9\ b)$$

$$x = x_0 + (\upsilon_0 \cos\theta_0)t \qquad\qquad y = y_0 + (\upsilon_0 \sin\theta)\,t - (½)\,g\,t^2 \qquad (4.9\ c)$$

Equation of trajectory:

$$y = y_0 + (\tan\theta)\,(x - x_0) - \frac{g}{2(\upsilon_0 \cos\ \theta_0)^2}\,(x - x_0)^2 \qquad (4.9\ d)$$

Note that these equations are more general than the ones discussed earlier. The initial coordinates are left unspecified as (x_0, y_0) rather than being placed at (0,0). Can you derive this general equation of the projectile trajectory? Do it before proceeding further?

Thus far you have studied motion of objects in a plane, which can be placed in the category of projectile motion. In projectile motion, the acceleration is constant both in magnitude and direction. There is another kind of two-dimensional motion in which acceleration is constant in magnitude but not in direction. This is uniform circular motion. Generally, circular motion refers to motion in a horizontal circle. However, motion in a vertical circle is also possible. You willl earn about them in the following section

Evangelista Torricelli
(1608 – 1647)

Italian mathematician and a student of Galelio Galili, he invented mercury barometer, investigated theory of projectiles, improved telescope and invented a primitive microscope. Disproved that nature abhors vacuum, presented torricellis theorem.

4.3 CIRCULAR MOTION

Look at Fig. 4.4a. It shows the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of a particle in uniform circular motion at two different instants of time t_1 and t_2, respectively. The word 'uniform' refers to constant speed. We have said that the speed of the particle is constant. What about its velocity? To find out velocity, recall the definition of average velocity and apply it to points P_1 and P_2:

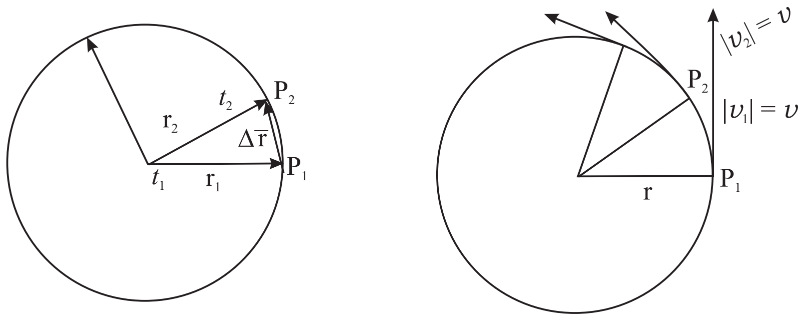
$$\boldsymbol{v}_{\mathbf{av}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1} = \frac{\Delta \mathbf{r}}{\Delta t} \qquad (4.10\text{ a})$$

The motion of a gramophone record, a grinding wheel at constant speed, the moving hands of an ordinary clock, a vehicle turning around a corner are examples of circular motion. The movement of gears, pulleys and wheels also involve circular motion. The simplest kind of circular motion is uniform circular motion. The most familiar example of uniform circular motion are a point on a rotating fan blade or a grinding wheel moving at constant speed.

One of the example of uniform circular motion is an artificial satellite in circular orbit around the earth. We have been benefitted immensely by the INSAT series of satellites and other artificial satellites. So let us now learn about uniform circular motion.

4.3.1 Uniform Circular Motion

By definition, ***uniform circular motion is motion with constant speed in a circle.***



(a) (b)

Fig. 4.4: (a) Positions of a particle in uniform circular motion; (b) Uniform circular motion

The vector $\Delta\mathbf{r}$ is shown in Fig. 4.4a. Now suppose you make the time interval Δt smaller and smaller so that it approaches zero. What happens to $\Delta\mathbf{r}$? In particular, what is the direction of $\Delta\mathbf{r}$? It approaches the tangent to the circle at point P_1 as Δt tends to zero. Mathematically, we define the instantaneous velocity at point P_1 as

$$\boldsymbol{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

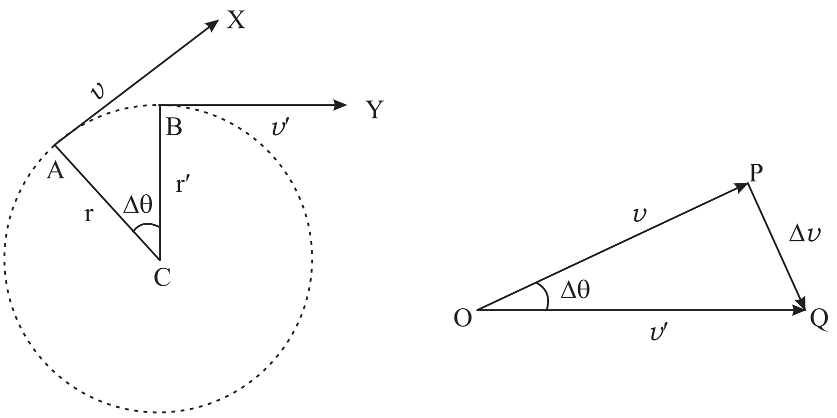
Thus, in uniform circular motion, the velocity vector changes continuously. Can you say why? This is because the direction of velocity is not constant. It goes on changing continuously as the particle travels around the circle (Fig. 4.4b). Because of ***this change in velocity, uniform circular motion is accelerated motion.*** The acceleration of a particle in uniform circular motion is termed as centripetal acceleration. Let us learn about it in some detail.

Centripetal acceleration : Consider a particle of mass m moving with a ***uniform speed*** υ in a circle. Suppose at any instant its position is at A and its motion is directed along AX. After a small time Δt, the particle reaches B and its velocity is represented by the tangent at B directed along BY.

Let $\mathbf{r}$ and $\mathbf{r}'$ be the position vectors and $\boldsymbol{\upsilon}$ and $\boldsymbol{\upsilon}'$; the velocities of the particle at A and B respectively as shown in Fig. 4.5 (a). The change in velocity $\Delta\boldsymbol{\upsilon}$ is obtained using the triangle law of vectors. As the path of the particle is circular and velocity is along its tangent, $\boldsymbol{\upsilon}$ is perpendicular to $\mathbf{r}$ and $\boldsymbol{\upsilon}'$ is perpendicular to $\Delta\mathbf{r}$. As the average acceleration $\left(\mathbf{a} = \dfrac{\Delta\boldsymbol{\upsilon}}{\Delta t}\right)$ is along $\Delta\boldsymbol{\upsilon}$, it (i.e., the average acceleration) is perpendicular to $\Delta\mathbf{r}$.

Let the angle between the position vectors $\mathbf{r}$ and $\mathbf{r}'$ be $\Delta\theta$. Then the angle between velocity vectors $\boldsymbol{\upsilon}$ and $\boldsymbol{\upsilon}'$ will also be $\Delta\theta$ as the velocity vectors are always perpendicular to the position vectors.

To determine the change in velocity $\Delta\boldsymbol{\upsilon}$ due to the change in direction, consider a point O outside the circle. Draw a line OP parallel to and equal to AX (or $\boldsymbol{\upsilon}$) and a line OQ parallel to and equal to BY (or $\boldsymbol{\upsilon}'$). As $|\boldsymbol{\upsilon}| = |\boldsymbol{\upsilon}'|$, OP = OQ. Join PQ. You get a triangle OPQ (Fig. 4.5b)



(a) (b)

Fig. 4.5

Now in triangle OPQ, sides OP and OQ represent velocity vectors $\boldsymbol{\upsilon}$ and $\boldsymbol{\upsilon}'$ at A and B respectively. Hence, their difference is represented by the side PQ in magnitude and direction. In other words the change in the velocity equal to PQ in magnitude and direction takes place as the particle moves from A to B in time Δt.

$\therefore$ Acceleration = Rate of change of velocity

$$\mathbf{a} = \frac{\mathbf{PQ}}{\Delta t} = \frac{\Delta \boldsymbol{v}}{\Delta t}$$

As Δt is very small AB is also very small and is nearly a straight line. Then Δ ACB and ΔPOQ are isosceles triangles having their included angles equal. The triangles are, therefore, similar and hence,

$$\frac{PQ}{AB} = \frac{OP}{CA}$$

or
$$\frac{\Delta v}{v.\Delta t} = \frac{v}{r}$$

[as magnitudes of velocity vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2 = v$ (say)]

or
$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

But $\frac{\Delta v}{\Delta t}$ is the acceleration of the particle. Hence

Centripetal acceleration, $a = \frac{v^2}{r}$

Since $v = r\,\omega$, the magnitude of centripetal force in given by

$$F = m\,a = \frac{m\,v^2}{r} = m\,r\omega^2.$$

As Δt is very small, $\Delta\theta$ is also very small and $\angle$OPQ = $\angle$OQP = 1 right angle.

Thus PQ is perpendicular to OP, which is parallel to the tangent AX at A. Now AC is also perpendicular to AX. Therefore AC is parallel to PQ. It shows that the contripetal force at any point acts towards the centre along the radius.

It shows that some minimum centripetal force has to be applied on a body to make it move in a circular path. In the absence of such a force, the body will move in a straight line path.

To experience this, you can perform a simple activity.

ACTIVITY 4.1

Take a small piece of stone and tie it to one end of a string. Hold the other end with your fingers and then try to whirl the stone in a horizontal or vertical circle. Start with a small speed of rotation and increase it gradually. What happens when the speed of rotation is low? Do you feel any pull on your fingers when the stone is whirling. What happens to the stone when you leave the end of the string you were holding? How do you explain this?

ACTIVITY 4.2

Take an aluminium channel of length one metre and bend it in the form shown in the diagram with a circular loop in the middle. Take help of some technical preson if required.

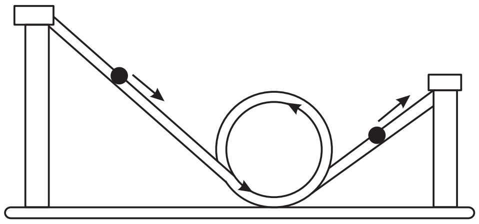


Fig. 4.6: The ball will loop if it starts rolling from a point high enough on the incline

Roll down a glass marble from different heights of the channel on the right hand side, and see whether the marble is able to loop the loop in each case or does it need some minimum height (hence velocity) below which the marble will not be able to complete the loop and fall down. How do you explain it?

Some Applications of Centripetal Force

(i) **Centrifuges :** These are spinning devices used for separating materials having different densities. When a mixture of two materials of different densities placed in a vessel is rotated at high speed, the centripetal force on the heavier material will be more. Therefore, it will move to outermost position in the vessel and hence can be separated. These devises are being used for uranium enrichment. In a chemistry laboratory these are used for chemical analysis.

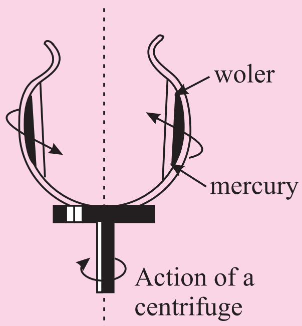


Fig. 4.7: When mercury and water are rotated in a dish, the water stays inside. Centripetal force, like gravitational force, is greater for the more dense substance.

(ii) Mud clings to an automobile tyre until the speed becomes too high and then it flies off tangentially (Fig. 4.8).

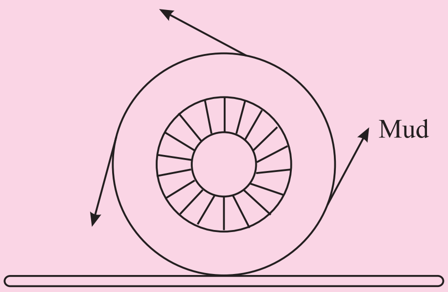


Fig. 4.8: Mud or water on a fast-turning wheel flies off tangentially

(iii) **Planetary motion :** The Earth and the other planets revolving round the sun get necessary centripetal force from the gravitational force between them and the sun.

Example 4.2 : Astronauts experience high acceleration in their flights in space. In the training centres for such situations, they are placed in a closed capsule, which is fixed at the end of a revolving arm of radius 15 m. The capsule is whirled around in a circular path, just like the way we whirl a stone tied to a string in a horizontal circle. If the arm revolves at a rate of 24 revolutions per minute, calculate the centripetal acceleration of the capsule.

Solution : The circumference of the circular path is $2\pi \times$ (radius) $= 2\pi \times 15$ m. Since the capsule makes 24 revolutions per minute or 60 s, the time it takes to go once around this circumference is $\frac{60}{24}$ s. Therefore,

speed of the capsule, $v = \frac{2\pi r}{T} = \frac{2\pi \times 15 \text{ m}}{(60/24) \text{ s}} = 38 \text{ ms}^{-1}$

The magnitude of the centripetal acceleration

$$a = \frac{v^2}{r} = \frac{(38 \text{ ms}^{-1})^2}{15 \text{ m}} = 96 \text{ ms}^{-2}$$

Note that centripetal acceleration is about 10 times the acceleration due to gravity.

4.3.2 Motion In a Vertical Circle

When a body moves in a horizontal circle, the direction of its linear velocity goes on changing but the angular velocity remains constant. But, when a body moves in a vertical circle, the angular velocity, too, cannot remain constant on account of the acceleration due to gravity.

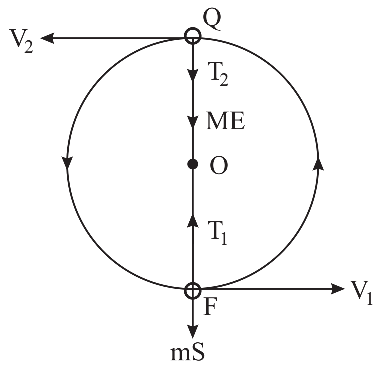


Fig. 4.9

Let a body of mass m tied to a string be rotated anticlockwise in a vertical circle of radius r about a point *O*. As the body rotates in the vertical circle, its speed is maximum at the lowest point *P*. It goes on decreasing as the body moves up to *Q*, and is minimum at the highest point *Q*. The speed goes on increasing as the body falls from *Q* to *P* along the circular path.

The forces acting on the body at *P* are weight of the body 'mg' and the tension T_1 of the string in the direction as shown in Fig. 4.9. Similarly, the forces acting on the body at *Q* are mg and the tension T_2 in the direction shown in Fig. 4.9. If v_1 and v_2 be the velocities of the body at *P* and *Q*, respectively, we have at *P*:

$$T_1 - mg = \frac{mv_1^2}{r}$$

or $$T_1 = \frac{mv_1^2}{r} + mg$$

Note that at *P*, the force ($T_1 - mg$) acts along *PO* and provides the centripetal force.

Similarly at *Q*,

$$T_2 + mg = \frac{mv_2^2}{r}$$

or $$T_2 = \frac{mv_2^2}{r} - mg$$

For the body to move along the circle without any slaking of the string,

$$T_2 \geq 0$$

i.e. the minimum value of the tension should be zero at *Q*.

When, $T_2 = 0$, $$mg = \frac{mv_2^2}{r}$$

i.e. the minimum velocity at the highest point of the circle is, $\sqrt{gr}$

$$\because \qquad \omega_2 = \frac{v}{r} = \sqrt{g/r}$$

The minimum velocity (v_1) at the lowest point (P) of the circle should be such that the velocity (v_2) at the highest point (Q) becomes $\sqrt{gr}$

Using the relation, $v^2 - u^2 = 2as$, we have

$$v_2^2 - v_1^2 = -2g(2r) \qquad (s = 2r \text{ and g is negative})$$

or
$$v_1^2 = v_2^2 + 4gr$$

$$v_1^2 = gr + 4gr = 5gr$$

or
$$v_1 = \sqrt{5gr}$$

Hence, for a body to go around a vertical circle completely minimum velocity at the lowest point should be $\sqrt{5gr}$.

or
$$w_1 = \sqrt{5g/r}$$

which shows that the angular velocity is also changing as the body moves in a vertical circle.

INTEXT QUESTIONS 4.2

1. In uniform circular motion, (a) Is the speed constant? (b) Is the velocity constant? (c) Is the magnitude of the acceleration constant? (d) Is acceleration constant? Explain.
2. In a vertical motion does the angular velocity of the body change? Explain.
3. An athlete runs around a circular track with a speed of 9.0 ms^{-1} and a centripetal acceleration of 3 ms^{-2}. What is the radius of the track?
4. The Fermi lab accelerator is one of the largest particle accelerators. In this accelerator, protons are forced to travel in an evacuated tube in a circular orbit of diameter 2.0 km at a speed which is nearly equal to 99.99995% of the speed of light. What is the centripetal acceleration of these protons?

   Take $c = 3 \times 10^8$ ms^{-1}.

4.4 APPLICATIONS OF UNIFORM CIRCULAR MOTION

So far you have studied that an object moving in a circle is accelerating. You have also studied Newton's laws in the previous lesson. From Newton's second law we can say that as the object in circular motion is accelerating, a net force must be acting on it.

What is the direction and magnitude of this force? This is what you will learn in this section. Then we will apply Newton's laws of motion to uniform circular motion. This helps us to explain why roads are banked, or why pilots feel pressed to their seats when they fly aircrafts in vertical loops.

Let us first determine the force acting on a particle that keeps it in uniform circular motion. Consider a particle moving with constant speed υ in a circle of radius r. From Newton's second law, the net external force acting on a particle is related to its acceleration by

$$\mathbf{F} = -\frac{m\upsilon^2}{r}\,\hat{\boldsymbol{r}}\,,\ |\mathbf{F}| = \frac{m\upsilon^2}{r} \tag{4.19}$$

This net external force directed towards the centre of the circle with magnitude given by Eqn. (4.19) is called ***centripetal force***. ***An important thing to understand and remember is that the term 'centripetal force' does not refer to a type of force of interaction like the force of gravitation or electrical force.*** This term only tells us that the net force of a certain magnitude acting on a particle in uniform circular motion is directed towards the centre. It does not tell us how this force is provided.

Thus, the force may be provided by the gravitational attraction between two bodies. For example, in the motion of a planet around the sun, the centripetal force is provided by the gravitational force between the two. Similarly, the centripetal force for a car travelling around a bend is provided by the force of friction between the road and the car's tyres and/or by the horizontal component of normal reaction of banked road. You will understand these ideas better when we apply them in certain concrete situations.

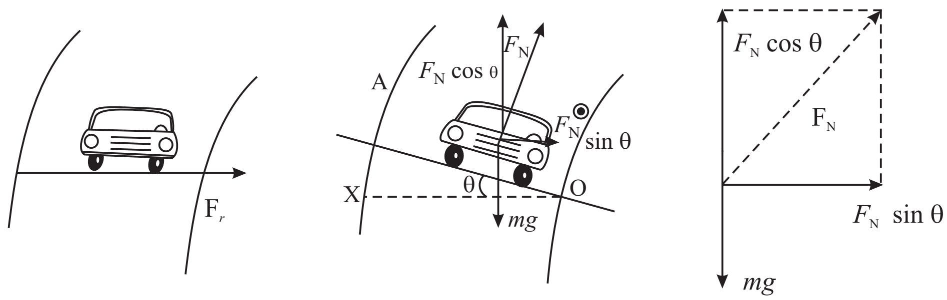
4.4.1 Banking of Roads

While riding a bicycle and taking a sharp turn, you may have felt that something is trying to throw you away from your path. Have you ever thought as to why does it happen?

You tend to be thrown out because enough centripetal force has not been provided to keep you in the circular path. Some force is provided by the friction between the tyres and the road, but that may not be sufficient. When you slow down, the needed centripetal force decreases and you manage to complete this turn.

Consider now a car of mass m, travelling with speed υ on a curved section of a highway (Fig. 4.10). To keep the car moving uniformly on the circular path, a force must act on it directed towards the centre of the circle and its magnitude must be equal to $m\upsilon^2/r$. Here r is the radius of curvature of the curved section.

Now if the road is levelled, the force of friction between the road and the tyres provides the necessary centripetal force to keep the car in circular path. This causes a lot of wear and tear in the tyre and may not be enough to give it a safe turn. The roads at curves are, therefore, banked, where banking means the raising of the outer edge of the road above the level of the inner edge (Fig. 4.10). As a matter of fact, roads are designed to minimise reliance on friction. For example, when car tyres are smooth or there is water or snow on roads, the coefficient of friction becomes negligible. Roads are banked at curves so that cars can keep on track even when friction is negligible.



(a) (b) (c)

Fig. 4.10 : A car taking a turn (a) on a level road; (b) on a banked road; and (c) Forces on the car with F_N resolved into its rectangular components. Generally θ is not as large as shown here in the diagram.

Let us now analyse the free body diagram for the car to obtain an expression for the angle of banking, θ, which is adjusted for the sharpness of the curve and the maximum allowed speed.

Consider the case when there is no frictional force acting between the car tyres and the road. The forces acting on the car are the car's weight mg and $\mathbf{F}_N$, the force of normal reaction. The centripetal force is provided by the horizontal component of $\mathbf{F}_N$. Thus, resolving the force $\mathbf{F}_N$ into its horizontal and vertical components, we can write

$$F_N \sin\theta = \frac{m v^2}{r} \qquad (4.20a)$$

Since there is no vertical acceleration, the vertical component of $\mathbf{F}_N$ is equal to the car's weight:

$$F_N \cos\theta = m\,g \qquad (4.20b)$$

We have two equations with two unknowns, i.e., F_N and θ. To determine θ, we eliminate F_N. Dividing Eqn. (4.20 a) by Eqn. (4.20 b), we get

$$\tan\theta = \frac{m v^2 / r}{m\,g} = \frac{v^2}{r\,g}$$

or

$$\theta = \tan^{-1}\frac{v^2}{rg} \qquad (4.21)$$

How do we interpret Eqn. (4.21) for limits on υ and choice of θ? Firstly, Eqn.(4.21) tells us that the angle of banking is independent of the mass of the vehicle. So even large trucks and other heavy vehicles can ply on banked roads.

Secondly, θ should be greater for high speeds and for sharp curves (i.e., for lower values of r). For a given θ, if the speed is more than υ, it will tend to move towards the outer edge of the curved road. So a vehicle driver must drive within prescribed speed limits on curves. Otherwise, the will be pushed off the road. Hence, there may be accidents.

Usually, due to frictional forces, there is a range of speeds on either side of υ. Vehicles can maintain a stable circular path around curves, if their speed remains within this range. To get a feel of actual numbers, consider a curved path of radius 300 m. Let the typical speed of a vehicle be 50 ms^{-1}. What should the angle of banking be? You may like to quickly use Eqn.(4.21) and calculate θ.

$$\theta = \tan^{-1} \frac{(50 \text{ ms}^{-1})^2}{(300 \text{ m})(9.8 \text{ ms}^{-2})} = \tan^{-1}(0.017) = 1^0$$

You may like to consider another application.

4.4.2 Aircrafts in vertical loops

On Republic Day and other shows by the Indian Air Force, you might have seen pilots flying aircrafts in loops (Fig. 4.11a). In such situations, at the bottom of the loop, the pilots feel as if they are being pressed to their seats by a force several times the force of gravity. Let us understand as to why this happens. Fig. 4.11b shows the 'free body' diagram for the pilot of mass m at the bottom of the loop.

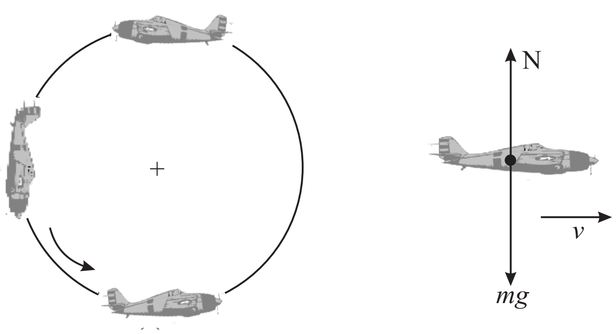


Fig. 4.11: (a) Aircrafts in vertical loops, (b) Free-body diagram for the pilot at the lowest point.

The forces acting on him are mg and the normal force N exerted by the seat. The net vertically upward force is N – mg and this provides the centripetal acceleration:

$$N - mg = m\,a$$

or $$N - mg = m\,v^2/r$$

or $$N = m\,(g + v^2/r)$$

In actual situations, if $v = 200\ \text{ms}^{-1}$ and $r = 1500$ m, we get

$$N = m\,g\left[1 + \frac{(200\ \text{m s}^{-1})^2}{(9.8\ \text{m s}^{-2} \times 1500\ \text{m})}\right] = m\,g \times 3.7$$

So the pilots feel as though force of gravity has been magnified by a factor of 3.7. If this force exceeds set limits, pilots may even black out for a while and it could be dangerous for them and for the aircraft.

INTEXT QUESTIONS 4.3

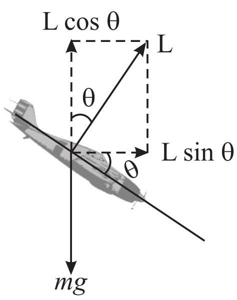


Fig. 4.12

1. Aircrafts usually bank while taking a turn when flying at a constant speed (Fig. 4.12). Explain why aircrafts do bank? Draw a free body diagram for this aircraft. ($\mathbf{F}_a$ is the force exerted by the air on the aircraft). Suppose an aircraft travelling at a speed $v = 100\ \text{ms}^{-1}$ makes a turn at a banking angle of 30^0. What is the radius of curvature of the turn? Take $g = 10\ \text{ms}^{-2}$.

2. Calculate the maximum speed of a car which makes a turn of radius 100 m on a horizontal road. The coefficient of friction between the tyres and the road is 0.90. Take $g = 10\ \text{ms}^{-2}$.

3. An interesting act performed at variety shows is to swing a bucket of water in a vertical circle such that water does not spill out while the bucket is inverted at the top of the circle. For this trick to be performed sucessfully, the speed of the bucket must be larger than a certain minimum value. Derive an expression for the minimum speed of the bucket at the top of the circle in terms of its radius R. Calculate the speed for R = 1.0 m.

WHAT YOU HAVE LEARNT

- **Projectile motion** is defined as the motion which has constant velocity in a certain direction and constant acceleration in a direction perpendicular to that of velocity:

$$a_x = 0 \qquad\qquad a_y = -g$$

$$v_x = v_0 \cos\theta \qquad v_y = v_0 \sin\theta - g\,t$$

$$x = x_0 + (v_0 \cos\theta)\,t \qquad y = y_0 + (v_0 \sin\theta) - ½ - g\,t^2$$

- Height $h = \dfrac{v_0^2 \sin 2\theta}{g}$
- Time of flight $T = \dfrac{2v_0 \sin\theta}{g}$
- Range of the projectile $R = \dfrac{v_0^2 \sin 2\theta}{g}$
- Equation of the Trajectory of a projectile $y = (\tan\theta_0)\,x - \dfrac{g}{2(v_0 \cos\theta_0)^2}\,x^2$
- **Circular motion** is uniform when the speed of the particle is constant. A particle undergoing ***uniform circular motion*** in a circle of radius r at constant speed v has a ***centripetal acceleration*** given by

  $$\mathbf{a}_r = -\frac{v^2}{r}\,\hat{\mathbf{r}}$$

  where $\hat{\mathbf{r}}$ is the unit vector directed from the centre of the circle to the particle. The speed v of the particle is related to its angular speed ω by $v = r\,\omega$.
- The ***centripetal force*** acting on the particle is given by

  $$\mathbf{F} = m\,\mathbf{a}_r = \frac{m\,v^2}{r}\,\hat{\mathbf{r}} = m\,r\,\omega^2$$

- When a body moves in a vertical circle, its angular velocity cannot remain constant.
- The minimum velocities at the highest and lowest points of a vetical circle are $\sqrt{gr}$ and $\sqrt{5gr}$ respectively

TERMINAL EXERCISE

1. Why does a cyclist bend inward while taking a turn on a circular path?
2. Explain why the outer rail is raised with respect to the inner rail on the curved portion of a railway track?
3. If a particle is having circular motion with constant speed, will its acceleration also be constant?
4. A stone is thrown from the window of a bus moving on horizontal road. What path will the stone follow while reaching the ground; as seen by a observer standing on the road?

5. A string can sustain a maximum force of 100 N without breaking. A mass of 1kg is tied to one end of the piece of string of 1m long and it is rotated in a horizontal plane. Compute the maximum speed with which the body can be rotated without breaking the string?
6. A motorcyclist passes a curve of radius 50 m with a speed of 10 m s^{-1}. What will be the centripetal acceleration when turning the curve?
7. A bullet is fired with an initial velocity 300 ms^{-1} at an angle of 30^0 with the horizontal. At what distance from the gun will the bullet strike the ground?
8. The length of the second's hand of a clock is 10 cm. What is the speed of the tip of this hand?
9. You must have seen actors in Hindi films jumping over huge gaps on horse backs and motor cycles. In this problem consider a daredevil motor cycle rider trying to cross a gap at a velocity of 100 km h^{-1}. (Fig. 4.13). Let the angle of incline on either side be 45^0. Calculate the widest gap he can cross.
10. A shell is fired at an angle of elevation of 30^0 with a velocity of 500 m s^{-1}. Calculate the vertical and horizontal components of the velocity, the maximum height that the shell reaches, and its range.
11. An aeroplane drops a food packet from a height of 2000 m above the ground while in horizontal flight at a constant speed of 200 kmh^{-1}. How long does the packet take to fall to the ground? How far ahead (horizontally) of the point of release does the packet land?

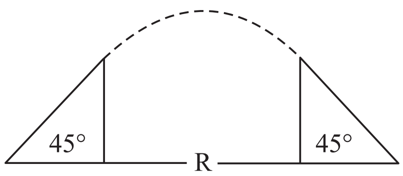


Fig. 4.13

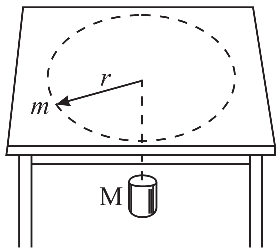


Fig. 4.14

12. A mass m moving in a circle at speed v on a frictionless table is attached to a hanging mass M by a string through a hole in the table (Fig. 4.14). Determine the speed of the mass m for which the mass M would remain at rest.
13. A car is rounding a curve of radius 200 m at a speed of 60 kmh^{-1}. What is the centripetal force on a passenger of mass m = 90 kg?

ANSWERS TO INTEXT QUESTIONS

4.1

(1) (a), (b), (d)

(2) (a) Yes (b) Yes (c) The ball with the maximum range.

(3) Maximum Range

$$\frac{v_0^2}{g} = \frac{(9.5\ \text{ms}^{-1})^2}{9.78\ \text{ms}^{-2}} = 9.23\ \text{m}$$

Thus, the difference is 9.23 m – 8.90 m = 0.33 m.

4.2

(1) (a) Yes (b) No (c) Yes (d) No

The velocity and acceleration are not constant because their directions are changing continuously.

(2) Yes. The angular velocity changes because of acceleration due to gravity

(3) Since

$$a = \frac{v^2}{r},\ r = \frac{v^2}{a} = \frac{(9.0\ \text{ms}^{-1})^2}{3\ \text{ms}^{-2}} = 27\ \text{m}$$

(4) $$a = \frac{c^2}{r} = \frac{(3\times10^8\ \text{ms}^{-1})^2}{10\times10^3\ \text{m}}$$

$= 9 \times 10^{13}\ \text{ms}^{-2}$

4.3

(1) This is similar to the case of banking of roads. If the aircraft banks, there is a component of the force L exerted by the air along the radius of the circle to provide the centripetal acceleration. Fig. 4.15 shows the free body diagram. The radius of curvature is

$$R = \frac{v^2}{g\tan\theta_0} = \left(\frac{100\ \text{ms}^{-1}}{10\ \text{ms}^{-2}\times\tan 30^\circ}\right)^2 = 10\sqrt{3}\ \text{m} = 17.3\ \text{m}$$

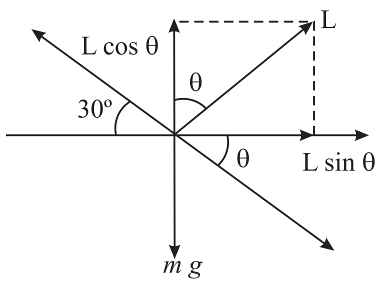


Fig. 4.15

(2) The force of friction provides the necessary centripetal acceleration :

$$F_s = \mu_s N = \frac{mv^2}{r}$$

Since the road is horizontal N – mg

$$\text{Thus} \qquad \mu_s\, mg = \frac{mv^2}{r}$$

$$\text{or} \qquad v^2 = \mu_s g\, r$$

$$\text{or} \qquad v = (0.9 \times 10 \text{ m s}^{-2} \times 100 \text{ m})^{1/2}$$

$$v = 30 \text{ ms}^{-1}.$$

(3) Refer to Fig. 4.16 showing the free body diagram for the bucket at the top of the circle. In order that water in the bucket does not spill but keeps moving in the circle, the force *mg* should provide the centripetal acceleration. At the top of the circle.

$$mg = \frac{mv^2}{r}$$

$$\text{or} \qquad v^2 = Rg$$

$$\therefore \qquad v = \sqrt{Rg}$$

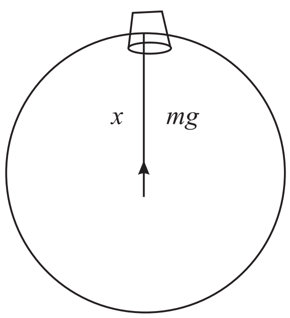


Fig. 4.16

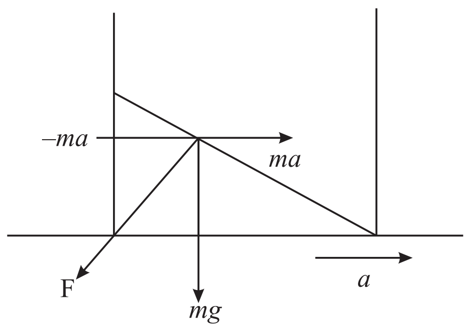


Fig. 4.17

This is the minimum value of the bucket's speed at the top of the vertical circle. For $R = 1.0$ m and taking $g = 10 \text{ ms}^{-2}$ we get

$v = 10 \text{ m s}^{-1} = 3.2 \text{ ms}^{-1}$

Answers to Terminal Problems

5. 10 ms^{-1}
6. 2 ms^{-2}
7. $900\sqrt{3}$ m

8. 1.05×10^{-3} ms^{-1}

9. 77.1 m

10. $\upsilon_x = 250\sqrt{3}$ ms^{-1}

    $\upsilon y = 250$ ms^{-1}

    Vertical height = 500 m

    Horizontal range = 3125 m

11. $t = 20$ s, 999.9 m

12. $\upsilon = \sqrt{\dfrac{m\,g\,r}{m}}$

13. 125 N