7

MOTION OF RIGID BODY

So far you have learnt about the motion of a single object, usually taken as a point mass. This simplification is quite useful for learning the laws of mechanics. But in real life, objects consist of very large number of particles. A tiny pebble contains millions of particles. Do we then write millions of equations, one for each particle? Or is there a simpler way? While discovering answer to this question you will learn about centre of mass and moment of inertia, which plays the same role in rotational motion as does mass in translational motion.

You will also study an important concept of physics, the angular momentum. If no external force acts on a rotating system, its angular momentum in conserved. This has very important implications in physics. It enables us to understand how a swimmer is able to somersault while diving from a diving board into the water below.

OBJECTIVES

After studying this lesson, you should be able to :

- *define the centre of mass of a rigid body;*
- *explain why motion of a rigid body is a combination of translational and rotational motions;*
- *define moment of inertia and state theorems of parallel and perpendicular axes;*
- *define torque and find the direction of rotation produced by it;*
- *write the equation of motion of a rigid body;*
- *state the principle of conservation of angular momentum; and*
- *calculate the velocity acquired by a rigid body at the end of its motion on an inclined plane.*

7.1 RIGID BODY

As mentioned earlier, point masses are ideal constructs, brought in for simplicity in discussion. In practice, when extended bodies interact with each other and the distances between them are very large compared to their sizes, their sizes can be ignored and they may be treated as point masses. *Can you give two examples of such cases where the sizes of the bodies are not important?* Sizes of stars are small as compared to the size of the galaxy. So, stars can be considered as point masses. Similarly, in the earth-moon system, moon's size can be ignored. But when we have to consider the rotation of a body about an axis, the size of the body becomes important. When we consider the rotation of a system, we generally assume that during rotation, the distances between its constituent particles remain fixed. Such a system of particles is called a ***rigid body***.

A rigid body is one in which the separation between the constituent particles does not change with its motion.

This definition implies that the shape of a rigid body is preserved during its motion. However, like a point mass, a rigid body is also an idealisation because, if we apply large forces, the distances between particles do change, may be infinitesimally. Therefore, in nature there is nothing like a perfectly rigid body. For most purposes, a solid body is good enough approximation to a rigid body. A cricket ball, a wooden block, a steel disc, even the earth and the moon could be considered as rigid bodies in this lesson.

Can water in a bucket be considered a rigid body? Obviously, water in a bucket cannot be a rigid body because it changes shape as bucket is pushed around.

You may now like to check what you have understood about a rigid body.

INTEXT QUESTIONS 7.1

1. A frame is made of six wooden rods. The rods are firmly attached to each other. Can this system be considered a rigid body?
2. Can a heap of sand be considered a rigid body? Explain your answer.

7.2 CENTRE OF MASS (C.M.) OF A RIGID BODY

Before we deal with rigid bodies consisting of several particles, let us consider a simpler case. Suppose we have a system of two particles having same mass joined by weightless and inextensible rod. *Can we consider this system as a rigid body?*

In this system, the distance between the two particles is fixed. So it is a rigid body.

Suppose that the two particles are at heights z_1 and z_2 from a horizontal surface (Fig. 7.1). Suppose further that the gravitational force is uniform in the small region in which the two particles move about. The force on each particle will be mg. The total force acting on the system is therefore $2mg$. We have now to find a point C somewhere in the system so that if a force $2mg$ acts at that point located at a height z from the horizontal surface, the motion of the system would be the same as with two forces. The potiential energies of particles 1 and 2 are mgz_1 and mgz_2, respectively. The potential energy of the particle at C is $2mgz$. Since this must be equal to the combined potential energy of the two particles, can write

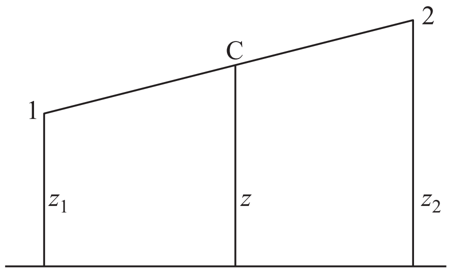


Fig. 7.1 : Two particle system

$$2\,mgz = mgz_1 + mgz_2 \tag{7.1}$$

or
$$z = \frac{z_1 + z_2}{2} \tag{7.2}$$

Note that the point C lies midway between the two particles. If the two masses were unequal, this point would not have been in the middle. If the mass of particle 1 is m_1 and that of particle 2 is m_2, Eqn. (7.1) modifies to

$$(m_1 + m_2)\,gz = m_1 gz_1 + m_2 gz_2 \tag{7.3}$$

so that
$$z = \frac{m_1 z_1 + m_2 z_2}{(m_1 + m_2)} \tag{7.4}$$

The point C is called the **centre of mass** (CM) of the system. As such, it is a mathematical tool and there is no physical point as CM.

To grasp this concept, study the following example carefully.

Example 7.1 : If in the above case, the mass of one particle is twice that of the other, let us locate the CM.

Solution : $m_1 = m$ and $m_2 = 2\,m$, Then Eqn. (7.4) gives

$$z = \frac{m\,z_1 + 2\,m\,z_2}{(m + 2\,m)} = \frac{z_1 + 2\,z_2}{3}$$

When a body consists of several particles, we generalise Eqn (7.4) to define its CM : *If the particle with mass m_1 has coordinates (x_1, y_1, z_1) with respect to some coordinate system, mass m_2 has coordinates (x_2, y_2, z_2) and so on* (*Fig.*7.2), *the coordinates of CM are given by*

$$x = \frac{m_1 x_1 + m_2 x_2 + \ldots}{m_1 + m_2 + \ldots}$$

$$= \frac{\sum_{i}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} \qquad (7.5)$$

$$= \frac{\sum_{i=1}^{N} m_i x_i}{M}$$

Similarly $$y = \frac{\sum_{i=1}^{N} m_i y_i}{M} \qquad (7.6)$$

and $$z = \frac{\sum_{i=1}^{N} m_i z_i}{M} \qquad (7.7)$$

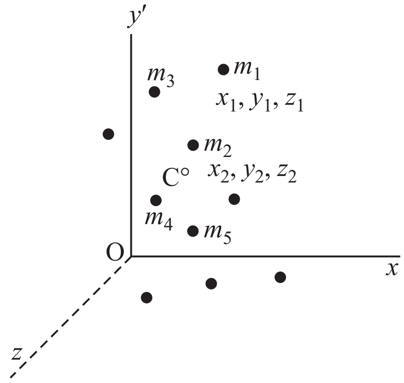


Fig. 7.2 : C.M. of a body consisting of several particles

where $\sum_{i=1}^{N} m_i$ denotes the sum over all the particles and, therefore, $\sum_{i=1}^{N} m_i$ is the total mass of the body, M.

Why should we define CM so precisely?

Recall that the rate of change of displacement is velocity, and the rate of change of velocity is acceleration. If a_{1x} denotes the component of acceleration of particle 1 along the x-axis and so on, from Eqn. (7.5), we can write

$$M a_x = m_1 a_{1x} + m_2 a_{2x} + \ldots \qquad (7.8)$$

where a_x is the acceleration of the centre of mass along x-axis. Similar equations can be written for accelerations along y- and z-axes. These equations can, however, be combined into a single equation using vector notation :

$$M \mathbf{a} = m_1 \mathbf{a_1} + m_2 \mathbf{a_2} + \ldots \qquad (7.9)$$

But the product of mass and acceleration is force. $m_1 \mathrm{a}_1$ is therefore the sum of all forces acting on particle 1. Similarly, $m_2 \mathbf{a_2}$ gives the net force acting on particle 2. The right hand side is, thus, the total force acting on the body.

The forces acting on a body can be of two kinds. Some forces can be due to sources outside the body. These forces are called the ***external*** forces. A familiar example is the force of gravity. Some other forces arise due to the interaction among the particles of the body. These are called ***internal*** forces. A familiar example is cohesive force.

In the case of a rigid body, the sum of the internal forces is zero because they cancel each other in pairs. Therefore, the acceleration of individual particles of the body are due to the sum or resultant of the external forces. In the light of this, we may write Eqn. (7.9) as

$$M\,\mathbf{a} = \mathbf{F}_{ext} \qquad (7.10)$$

This shows that ***the CM of a body moves as though the entire mass of the body were located at that point and it was acted upon by the sum of all the external forces acting on the body.*** Note the simplification introduced in the derivation by defining the centre of mass. We donot have to deal with millions of individual particles now, only the centre of mass needs to be located to determine the motion of the given body. The fact that the motion of the CM is determined by the external forces and that the internal forces have no role in this at all leads to very interesting consequences.

You are familiar with the motion of a projectile. *Can you recall what path is traced by a projectile?*

The motion is along a parabolic path. Suppose the projectile is a bomb which explodes in mid air and breaks up into several fragments. The explosion is caused by the internal forces. There is no change in the external force, which is the force of gravity. The centre of mass of the projectile, therefore, continues to be the same parabola on which the bomb would have moved if it had not exploded (Fig. 7.3). The fragments may fly in all directions on different parabolic paths but the centre of mass of the various fragments will lie on the original parabola.

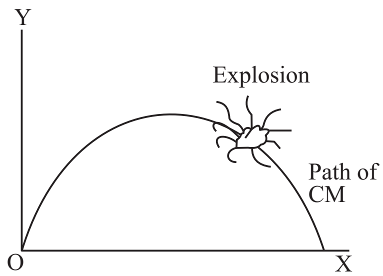


Fig. 7.3 : Centre of mass of a projectile

You might have now understood the importance of the concept of centre of mass of a rigid body. You will encounter more examples of importance in subsequent sections. Let us therefore see how the centre of mass of a system is obtained by taking a simple example.

Example 7.2 : Suppose four masses, 1.0 kg, 2.0 kg, 3.0 kg and 4.0 kg are located at the corners of a square of side 1.0 m. Locate its centre of mass?

Solution : We can always make the square lie in a plane. Let this plane be the (x,y) plane. Further, let us assume that one of the corners coincides with the origin of

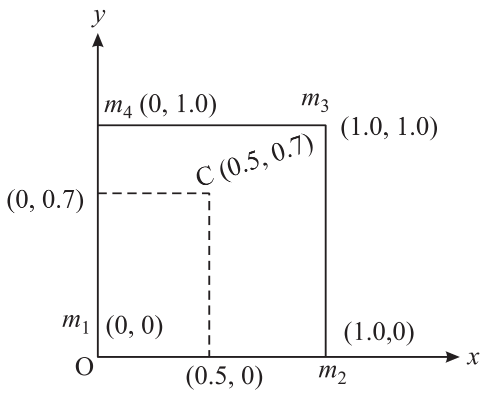


Fig. 7.4 : Locating CM of four masses placed at the corners of a square

the coordinate system and the sides are along the x and y axes. The coordinates of the four masses are : m_1 (0,0), m_2 (1.0,0), m_3 (1.0,1.0) and m_4 (0, 1.0), where all distances are expressed in metres (Fig.7.4).

From Eqns. (7.5) and (7.6), we get

$$x = \frac{1.0 \times 0 + 2.0 \times 1.0 + 3.0 \times 1.0 + 4.0 \times 0}{1.0 + 2.0 + 3.0 + 4.0} \text{ m}$$

$$= 0.5 \text{ m}$$

and

$$y = \frac{1.0 \times 0 + 2.0 \times 0 + 3.0 \times 1.0 + 4.0 \times 1.0}{1.0 + 2.0 + 3.0 + 4.0} \text{ m}$$

$$= 0.7 \text{ m}$$

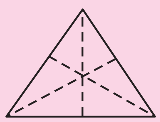
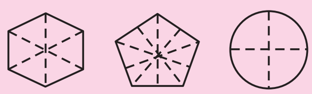
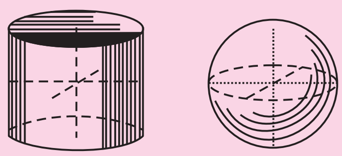
The CM has coordinates (0.5 m, 0.7 m) and is marked C in Fig.7.4. Note that the CM is not at the centre of the square although the square is a symmetrical figure.

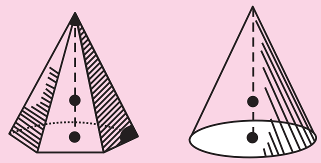
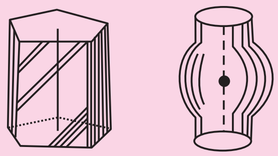
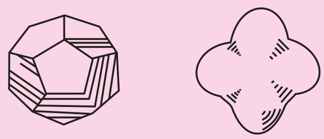
What could be the reason for the CM not being at the centre? To discover answer to this question, calculate the coordinates of CM if all masses are equal.

7.2.1 CM of Some Bodies

The position of centre of mass of extended bodies can not be easily calculated because a very large number of particles constituting the body have to be considered. The fact that all the particles of a rigid body have same mass and are uniformly distributed makes things somewhat simpler. If the body is regular in shape and possesses some symmetry, say it is cylindrical or spherical, the calculation is a little bit simplified. But even such calculations are beyond the scope of this course. However, keeping in mind the importance of CM, we give in Table 7.1 the position of CM of some regular, symmetrical bodies.

Table 7.1 Centres of Mass of some regular symmetrical bodies

Figure	Position of Centre of Mass
	*Triangular plate* Point of intersection of the three medians
	*Regular polygon and circular plate* At the geometrical centre of the figure
	*Cylinder and sphere* At the geometrical centre of the figure

	*Pyramid and cone* On line joining vertex with centre of base and at h/4 of the height measured from the base.
	*Figure with axial symmetry* Some point on the axis of symmetry
	*Figure with centre of symmetry* At the centre of symmetry

Two things must be remembered about the centre of mass : (i) It may be outside the body as in case of a ring. (ii) When two bodies revolve around each other, they actually revolve around their common centre of mass. For example, stars in a binary system revolve around their common centre of mass. The Earth-Sun system also revolves around its common centre of mass. But since mass of the Sun is very large as compared to the mass of earth, the centre of mass of the system is very close to the centre of the Sun.

Now it is time to check your progress.

INTEXT QUESTIONS 7.2

1. The grid shown here has particles A, B, C, D and E respectively have masses 1.0 kg, 2.0kg, 3.0 kg, 4.0 kg and 5.0 kg. Calculate the coordinates of the position of the centre of mass of the system (Fig. 7.5).

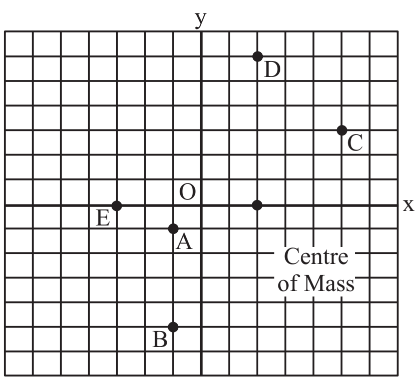


Fig. 7.5

2. If three particles of masses m_1 =1 kg, m_2 = 2 kg, and m_3 = 3 kg are situated at the corners of an equilateral triangle of side 1.0 m, obtain the position coordinates of the centre of mass of the system.

3. Show that the ratio of the distances of the two particles from their common centre of mass is inversely proportional to the ratio of their masses.

7.3 TRANSLATIONAL AND ROTATIONAL MOTION OF A RIGID BODY : A COMPARISON

When a rigid body moves in such a way that all its particles move along parallel paths (Fig.7.6), its motion is called **translational** motion. Since all the particles execute identical motion, its centre of mass must also be tracing out an identical path. Therefore, the translational motion of a body may be characterised by the motion of its centre of mass. We have seen that this motion is given by Eqn.(7.10):

$$M\,\mathbf{a} = \mathbf{F}_{\text{ext}}$$

Do you now see the advantage of defining the centre of mass of a body? With its help, the translational motion of body can be described by an equation for a single particle having mass equal to the mass of the whole body. It is located at the centre of mass and is acted upon by the sum of all the external forces which are acting on the rigid body. To understand the concept clearly, perform the following activities.

ACTIVITY 7.1

Take a wooden block. Make two or three marks on any of its surfaces. Now keep the marked surface in front of you and push the block along a horizontal floor. Note the paths traced by the marks. All these marks have paths parallel to the floor and, therefore, parallel to one another (Fig. 7.6). You can easily see that the lengths of the paths are also equal.

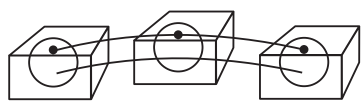


Fig. 7.6 : A wooden block moving along the floor performs translational motion.

ACTIVITY 7.2

Let us now perform another simple experiment. Take a cylindrical piece of wood. On its plane face make a mark or two. Now roll the cylinder slowly on the floor, keeping the plane face towards you. You would notice that the mark such as A in Fig. 7.7, has not only moved parallel to the floor, but has also performed circular motion. So, the body has performed both translational and rotational motion.

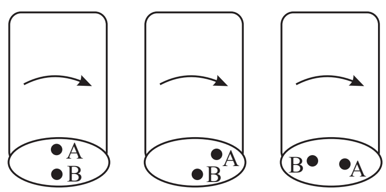


Fig. 7.7 : Rolling motion of a cylinder: The point A has not only moved parallel to the floor but also performed circular motion

While the general motion of a rigid body consists of both translation and rotation, it cannot have translational motion if one point in the body is fixed; it can then only rotate. The most convenient point to fix for this purpose is the CM of the body.

You might have seen a grinding stone (the chakki). The handle of the stone moves in a circular path. All the points on the stone also move in circular paths around an axis passing through the centre of the stone (Fig.7.8).

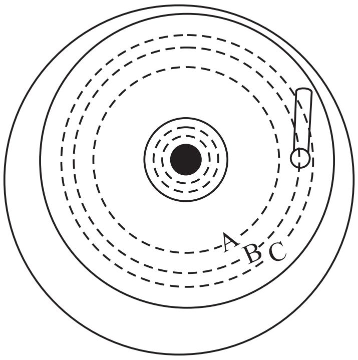


Fig. 7.8 : Pure rotation of a grinding stone

The motion of a rigid body in which all its constituent particles describe concentric circular paths is known as rotational motion.

We have noted above that translational motion of a rigid body can be described by an equation similar to that of a single particle. You are familiar with such equations. Therefore, in this lesson we concentrate only on the rotational motion of a rigid body. The rotational motion can be obtained by keeping a point of the body fixed so that it cannot have any translational motion. For the sake of mathematical convenience, this point is taken to be the CM. The rotation is then about an axis passing through the CM. A good example of rotational motion is the earth's rotation about its own axis (Fig. 7.9). You have studied in earlier lessons that the mass of the body plays a very important role. It determines the acceleration acquired by the body for a given force. Can we define a similar quantity for rotational motion also? Let us find out.

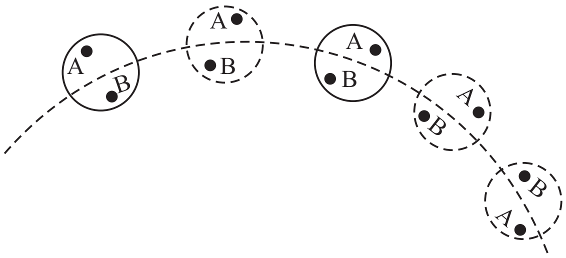


Fig. 7.9 : Rotation of the earth

7.3.1 Moment of Inertia

Let C be the centre of mass of a rigid body. Suppose it rotates about an axis through this point (Fig.7.10).

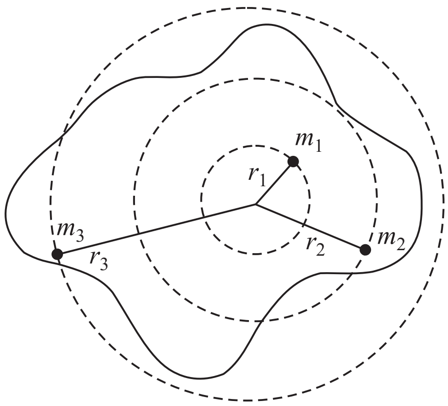


Fig. 7.10 : Rotation of a plane lamina about an axis passing through its centre of mass

Suppose particles of masses m_1, m_2, m_3...are located at distances r_1, r_2, r_3...from the axis of rotation and are moving with speeds v_1,

υ_2, υ_3 respectively. Then particle 1 has kinetic energy (½) $m_1 \upsilon_1^2$. Similarly, the kinetic energy of particle of mass m_2 is (½) $m_2 \upsilon_2^2$. By adding the kinetic energies of all the particles, we get the total energy of the body. If T denotes the total kinetic energy of the body, we can write

$$T = (½)\, m_1 \upsilon_1^2 + (½)\, m_2 \upsilon_2^2 + ...$$

$$= \sum_{i=1}^{i=1} \left(\frac{1}{2}\right) m_i \upsilon_i^2 \qquad (7.11)$$

where $\sum_{i=1}^{i=n}$ indicates the sum over all the particles of the body.

You have studied in lesson 4 that angular speed (ω) is related to linear speed (υ) through the equation $\upsilon = r\,\omega$. Using this result in Eqn. (7.11), we get

$$T = \sum_{i=1}^{i=n} \left(\frac{1}{2}\right) m_i (r_i \omega)^2 \qquad (7.12)$$

Note that we have not put the subscript i with w because all the particles of a rigid body have the same angular speed. Eqn. (7.12) can now be rewritten as

$$T = \frac{1}{2}\left(\sum_{i=1}^{i=n} m_i\, r_i^2\right)\omega^2$$

$$= \frac{1}{2}\, \mathrm{I}\, \omega^2 \qquad (7.13)$$

The quantity $$I = \sum_{i} m_i\, r_i^2 \qquad (7.14)$$

is called the ***moment of inertia*** of the body.

Example 7.3 : *F*our particles of mass *m* each are located at the corners of a square of side *L*. Calculate their moment of inertia about an-axis passing through the centre of the square and perpendicular to its plane.

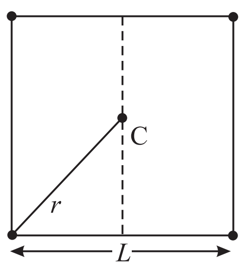


Fig. 7.11

Solution : Simple geometry tells us that the distance of each particle from the axis of rotation is $r = L\sqrt{2}$. Therefore, we can write

$$I = m\,r^2 + m\,r^2 + m\,r^2 + m\,r^2$$

$$= 4m\,r^2$$

$$= 4\,m\left(\frac{L}{\sqrt{2}}\right)^2 \qquad \left(\text{Since } r = \frac{L}{\sqrt{2}}\right).$$

$$= 2\,m\,L^2$$

It is important to remember that ***moment of inertia*** is defined with reference to an axis of rotation. Therefore, whenever you mention moment of inertia, the axis of rotation must also be specified. In the present case, I is the moment of inertia about an axis passing through the centre of the square and normal to the plane containing four perfect masses. (Fig. 7.10) The moment of inertia is expressed in kg m^2.

The moment of inertia of a rigid body is often written as

$$I = M\,K^2 \qquad (7.15)$$

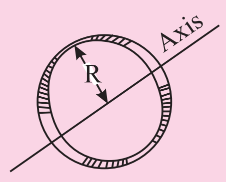
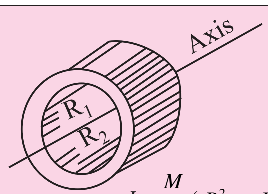
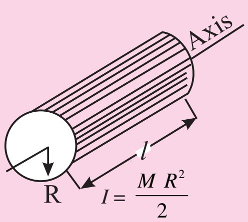
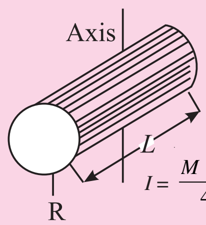
where M is the total mass of the body and K is called the ***radius of gyration*** of the body. ***The radius of gyration is that distance from the axis of rotation where the whole mass of the body can be assumed to be placed to get the same moment of inertia which the body actually has.*** It is important to remember that the moment of inertia of a body about an axis depends on the distribution of mass around that axis. If the distribution of mass changes, the moment of inertia will also change. This can be easily seen from Example 7.3. Suppose we place additional masses at one pair of opposite corners of amount m each. Then the moment of inertia of the system about the axis through C and perpendicular to the plane of square would be

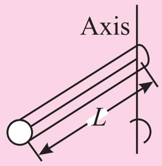
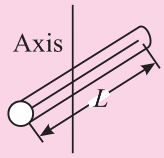
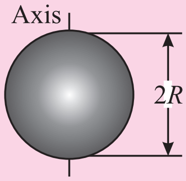
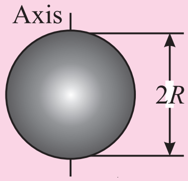
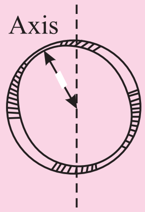
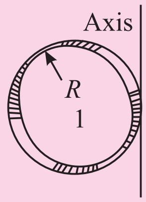
$$\text{I} = m\,r^2 + 2m\,r^2 + m\,r^2 + 2m\,r^2$$

$$= 6m\,r^2$$

Note that moment of inertia has changed from $2mL^2$ to $3\,mL^2$.

Table 7.2 Moments of inertia of a few regular and uniform bodies.

 Hoop about central axis $I = MR^2$	 Annular cylinder (or ring) about cylinder axis $I = \frac{M}{2}(R_1^2 + R_2^2)$
 Solid cylinder about cylindrical axis $I = \frac{M\,R^2}{2}$	 Solid cylinder (or disk) about a central diameter $I = \frac{M\,R^2}{4} + \frac{M^2\,\ell^2}{12}$

 $I = \frac{ML^2}{12}$	Thin rod about an axis passing through its centre and normal to its length	 $I = \frac{ML^2}{3}$	Thin rod about an axis passing through one end and perpendicular to length
 $I = \frac{2MR^2}{5}$	Solid sphere about any diameter	 $I = \frac{2MR^2}{3}$	Thin spherical shell about any diameter
 $I = \frac{MR^2}{2}$	Hoop about any diameter	 $I = \frac{3MR^2}{2}$	Hoop about any tangent line

Refer to Eqn.(7.13) again and compare it with the equation for kinetic energy of a body in linear motion. Can you draw any analogy? You will note that in rotational motion, the role of mass has been taken over by the moment of inertia and the angular speed has replaced the linear speed.

A. Physical significance of moment of inertia

The physical significance of moment of inertia is that it performs the same role in rotational motion that the mass does in linear motion.

Just as the mass of a body resists change in its state of linear motion, the moment of inertia resists a change in its rotational motion. This property of the moment of inertia has been put to a great practical use. Most machines, which produce rotational motion have as one of their components a disc which has a very large moment of inertia. Examples of such machines are the steam engine and the automobile engine. The disc with a large moment of inertia is called a ***flywheel***. To understand how a flywheel works, imagine that the driver of the engine wants to suddenly increase the speed. Because of its large moment of inertia, the flywheel resists this attempt. It allows only a gradual increase in speed. Similarly, it works against the attempts to suddenly reduce the speed, and allows only a gradual decrease in the speed. Thus , the flywheel, with its large moment of inertia, prevents jerky motion and ensures a smooth ride for the passengers.

We have seen that in rotational motion, angular velocity is analogous to linear velocity in linear motion. Since angular acceleration (denoted usually by α) is the rate of change of angular velocity, it must correspond to acceleration in linear motion.

B. Equations of motion for a uniformly rotating rigid body

Consider a lamina rotating about an axis passing through O and normal to its plane. If it is rotating with a constant angular velocity ω, as shown, then it will turn through an angle θ in t seconds such that

$$\theta = \omega t \qquad 7.16(a)$$

However, if the lamina is subjected to constant torque (which is the turning effect of force), it will undergo a constant angular acceleration. The following equations describe its rotational motion:

$$\omega_f = \omega_i + \alpha t \qquad 7.16(b)$$

where ω_i is initial angular velocity and ω_f is final angular velocity.

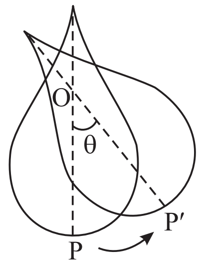


Fig. 7.12 : Rotation of a lamina about a fixed nail

Similarly, we can write

$$\theta = \omega_i t + \frac{1}{2} \alpha t^2 \qquad 7.16(c)$$

$$\omega_f^2 = \omega_i^2 + 2 \alpha \theta \qquad 7.16(d)$$

where θ is angular dispalcement in t seconds.

On a little careful scrutiny, you can recognise the similarity of these equations with the corresponding equations of kinematics for translatory motion.

Example 7.4 : A wheel of a bicycle is free to rotate about a horizontal axis (Fig. 7.11). It is initially at rest. Imagine a line OP drawn on it. By what angle would the line OP move in 2 s if it had a uniform angular acceleration of 2.5 rad s^{-2}.

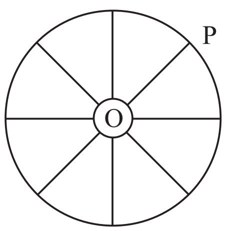


Fig. 7.13 : Rotation of bicycle wheel

Solution : Angular displacement of line OP is given by

$$\theta = \omega_0 t + (½)\, \alpha\, t^2$$

$$= 0 + (½) \times (2.5 \text{ rad s}^{-2}). \times 4 \text{ s}^2$$

$$= 5 \text{ rad}$$

We have mentioned above that for rotational motion of a rigid body, its CM is kept fixed. However, it is just a matter of convenience that we keep CM fixed. But many a time, we consider points other than the CM. That is, a point in the body which can also be kept fixed and the body rotated about it. But then the axis of rotation will pass through this fixed point. The moment of inertia about this axis would be different from the moment of inertia about an axis passing through the CM. The relation between the two moments of inertia can be obtained using the theorems of moment of inertia.

7.3.2 Theorems of moment of inertia

There are two theorems which connect moments of inertia about two axes; one of which is passing through the CM of the body. These are :

(i) the theorem of parallel axes, and

(ii) the theorem of perpendicular axes.

Let us now learn about these theorems and their applications.

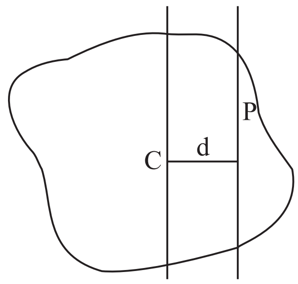


Fig. 7.14 : Parallel axes through CM and another point P

(*i*) Theorem of parallel axes

Suppose the given rigid body rotates about an axis passing through any point P other than the centre of mass. The moment of inertia about this axis can be found from a knowledge of the moment of inertia about a parallel axis through the centre of mass. Theorem of parallel axis states that *the moment of inertia about an axis parallel to the axis passing through its centre of mass is equal to the moment of inertia about its centre of mass plus the product of mass and square of the perpendicular distance between the parallel axes*. If I denotes the required moment of inertia and I_C denotes the moment of inertia about a parallel axis passing through the CM, then

$$I = I_C + M\, d^2 \qquad (7.17)$$

where M is the mass of the body and d is the distance between the two axes (Fig. 7.12). This is known as the **theorem of parallel axes.**

(*ii*) Theorem of perpendicular axes

Let us choose three mutually perpendicular axes, two of which, say *x* and *y* are in the plane of the body, and the third, the *z*–axis, is perpendicular to the plane (Fig.7.13). The perpendicular axes theorem states that ***the sum of the moments of inertia about x and y axes is equal to the moment of inertia about the z–axis.***

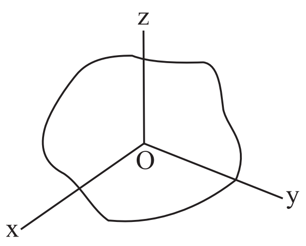


Fig. 7.15 : Theorem of perpendicular axes

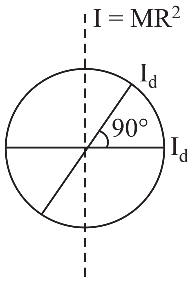


Fig. 7.16 : Moment of inertia of a hoop

That is,

$$I_z = I_x + I_y \tag{7.18}$$

We now illustrate the use of these theorems by the following example.

Let us take a hoop shown in Fig. 7.16. From Table 7.2 you would recall that moment of inertia of a hoop about an axis passing through its centre and perpendicular to the base is $M R^2$, where M is its mass and R is its radius. The theorem of perpendicular axes tells us that this must be equal to the sum of the moments of inertia about two diameters which are perpendicular to each other as well as to the central axis. The symmetry of the hoop tells us that the moment of inertia about any diameter is the same as about any other diameter. This means that all the diameters are **equivalent** and any two perpendicular diameters may be chosen. Since the moment of inertia about each is the same, say I_d, Eqn.(7.18) gives

$$M R^2 = 2 I_d$$

and therefore

$$I_d = (½) MR^2$$

So, the moment of inertia of a hoop about any of its diameter is (½) $M R^2$.

Let us now take a point P on the rim. Consider a tangent to the hoop at this point which is parallel to the axis of the hoop. The distance between the two axes is obviously equal to R. The moment of inertia about the tangent can be calculated using the theorem of parallel axes. It is given by

$$I_{\tan} = M R^2 + M R^2 = 2 M R^2.$$

It must be mentioned that many of the entries in Table 7.2 have been computed using the theorems of parallel and perpendicular axes.

Notes

7.3.3 Torque and Couple

ACTIVITY 7.3

Have you ever noticed that it is easy to open the door by applying force at a point far away from the hinges? What happens if you try to open a door by applying force near the hinges? Carry out this activity a few times. You would realise that much more effort is needed to open the door if you apply force near the hinges than at a point away from the hinges. Why is it so? Similarly, for turning a screw we use a spanner with a long handle. What is the purpose of keeping a long handle? Let us seek answers to these questions now.

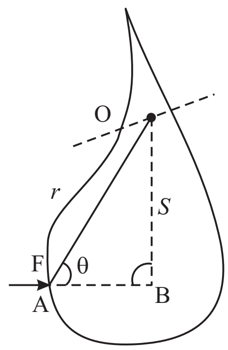


Fig. 7.17 : Rotation of a body

Suppose O is a fixed point in the body and it can rotate about an axis passing through this point (Fig.7.17). Let a force of magnitude F be applied at the point A along the line AB. If AB passes through the point O, the force F will not be able to rotate the body. The farther is the line AB from O, the greater is the ability of the force to turn the body about the axis through O. The ***turning effect of a force is called torque.*** Its magnitude is given by

$$\tau = F\,s = F\,r\sin\theta \qquad (7.19)$$

where s is the distance between the axis of rotation and the line along which the force is applied.

The units of torque are newton-metre, or Nm. The torque is actually a vector quantity. The vector from of Eqn.(7.19) is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \qquad (7.20)$$

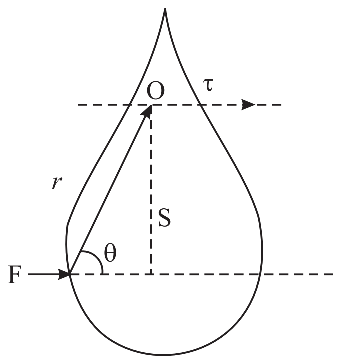


Fig. 7.18 : Right hand thumb rule

which gives both magnitude and direction of the torque. What is the direction in which the body would turn? To discover this, we recall the rules of vector product (refer to lesson 1) : τ is perpendicular to the plane containing vectors **r** and **F**, which is the plane of paper here (Fig.7.18). If we extend the thumb of the right hand at right angles to the fingers and curl the fingers so as to point from **r** to **F** through the smaller angle, the direction in which thumb points is the direction of $\boldsymbol{\tau}$.

Apply the above rule and show that the turning effect of the force in Fig. 7.18 is normal to the plane of paper downwards. This corresponds to clockwise rotation of the body.

Example 7.5 : Fig.7.19 shows a bicycle pedal. Suppose your foot is at the top and you are pressing the pedal downwards. (*i*) What torque do you produce? (ii) Where should your foot be for generating maximum torque?

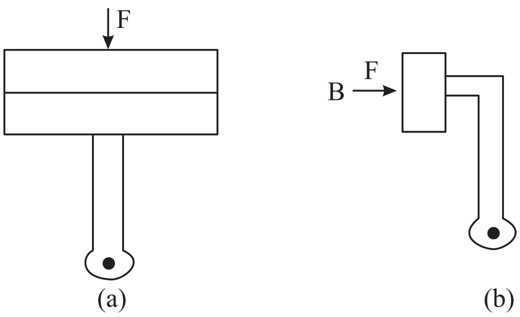


Fig. 7.19 : A bicycle pedal (a) at the top when $\tau = 0$; (b) when τ is maximum

Solution : (i) When your foot is at the top, the line of action of the force passes through the centre of the pedal. So, $\theta = 0$, and $\tau = Fr\sin\theta = 0$.

(ii) To get maximum torque, $\sin\theta$ must have its maximum value, that is θ must be 90°. This happens when your foot is at position B and you are pressing the pedal downwards.

If there are several torques acting on a body, the net torque is the vector sum of all the torques. Do you see any correspondence between the role of torque in the rotational motion and the role of force in the linear motion? Consider two forces of equal magnitude acting on a body in opposite directions (Fig.7.20). Assume that the body is free to rotate about an axis passing through O. The two torques on the body have magnitudes

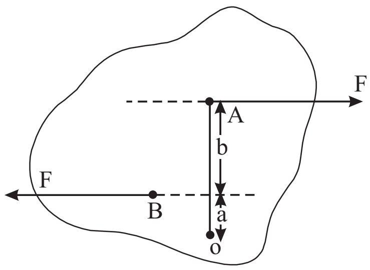


Fig. 7.20 : Two opposite forces acting on body

$$\boldsymbol{\tau}_1 = (a + b)\, F$$

and $$\boldsymbol{\tau}_2 = a\, F.$$

You can verify that the turning effect of these torques are in the opposite directions. Therefore, the magnitude of the net turning effect on the body is in the direction of the larger torque, which in this case is $\boldsymbol{\tau}_1$:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 = \mathbf{bF} \qquad (7.21)$$

We may therefore conclude that ***two equal and opposite forces having different lines of action are said to form a couple whose torque is equal to the product of one of the forces and the perpendicular distance between them.***

There is another useful expression for torque which clarifies its correspondence with force in linear motion. Consider a rigid body rotating about an axis passing through a point O (Fig. 7.21). Obviously, a particle like P is rotating about the axis in a circle of radius r. If the circular motion is non–uniform, the particle experiences forces in the radial direction as well as in the tangential direction. The radial force is the centripetal force $m\,\omega^2 r$, which keeps the particle in the circular path. The tangential force is required to change the magnitude of υ, which at every instant is along the tangent to the circular path. Its magnitude is $m\,a$, where a is the tangential acceleration. ***The radial force does not produce any torque.*** *Do you know why? The tangential force produces a torque of magnitude m a r.* Since $a = r\,\alpha$, where α is the angular acceleration, the magnitude of the torque is $m\,r^2\,\alpha$. If we consider all the particles of the body, we can write

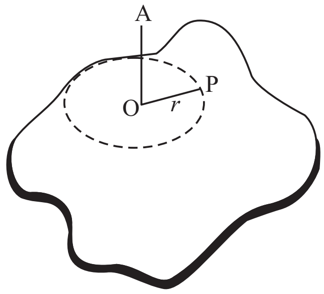


Fig. 7.21 : A rigid body rotating about on axis

$$\tau = \sum_{i=0}^{i=n} m_i\, r_i^2\, \alpha = \left(\sum_i m_i\, r_i^2\right)\alpha$$

$$= \mathrm{I}\,\alpha. \qquad (7.22)$$

because α is same for all the particles at a given instant.

The similarity between this equation and $\mathbf{F} = m\,a$ shows that $\boldsymbol{\tau}$ performs the same role in rotational motion as $\mathbf{F}$ does in linear motion. A list of corresponding quantities in rotational motion and linear motion is given in Table 7.3. With the help of this table, you can write any equation for rotational motion if you know its corresponding equation in linear motion.

Table 7.3 : Corresponding quantities in rotational and translational motions

Translational Motion		**Rotation about a Fixed Axis**	
Displacement	x	Angular displacement	θ
Velocity	$\upsilon = \dfrac{dx}{dt}$	Angular velocity	$\omega = \dfrac{d\theta}{dt}$
Acceleration	$a = \dfrac{dv}{dt}$	Angular acceleration	$\alpha = \dfrac{d\omega}{dt}$
Mass	M	Moment of inertia	I
Force	$F = m\,a$	Torque	$\tau = I\,\alpha$
Work	$W = \int \mathrm{F}\,dx$	Work	$W = \int \tau\, d\theta$
Kinetic energy	$\frac{1}{2}M\,\upsilon^2$	Kinetic energy	$(\frac{1}{2})\,I\,\omega^2$
Power	$\mathrm{P} = \mathrm{F}\upsilon$	Power	$P = \tau\,\omega$
Linear momentum	$M\,\upsilon$	Angular momentum	$I\,\omega$

With the help of Eqn.(7.22) we can calculate the angular acceleration produced in a body by a given torque.

Example 7.6 : A uniform disc of mass 1.0 kg and radius 0.1m can rotate about an axis passing through its centre and normal to its plane without friction. A massless string goes round the rim of the disc and a mass of 0.1 kg hangs at its end (Fig.7.22). Calculate (i) the angular acceleration of the disc, (ii) the angle through which the disc rotates in one second, and (iii) the angular velocity of the disc after one second. Take $g = 10\ \text{ms}^{-2}$

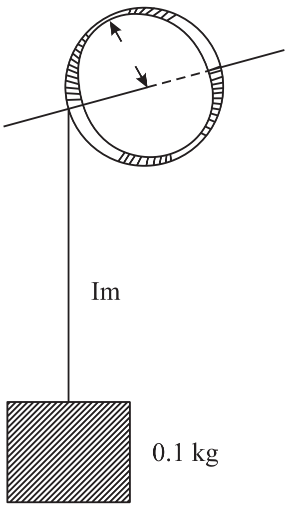


Fig. 7.22

Solution : (i) If R and M denote the radius and mass of the disc, from Table 7.2, we recall that its moment of inertia is given by $I = (½)\, M R^2$. If F denotes the magnitude of force (= $m\, g$) due to the mass at the end of the string then $\tau = F\, R$. Eqn. (7.22) now gives

$$\alpha = \tau / I = FR/I = 2F/MR$$

$$= \frac{2 \times (0.1\text{kg}) \times (10\text{ms}^{-2})}{(1.0\,\text{kg}) \times (0.1\,\text{m})} = 20\ \text{rad s}^{-2}.$$

(ii) For angle θ through which the disc rotates, we use Eqn.(7.16). Since the initial angular velocity is zero, we have

$$\theta = (½) \times 20 \times 1.0 = 10\ \text{rad}$$

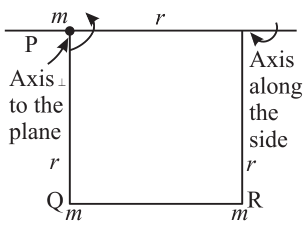
(iii) For the velocity after one second, we have

$$\omega = \alpha\, t = 20 \times 1.0 = 20\ \text{rad s}^{-1}$$

Now, you may like to check your progress.Try the following questions.

INTEXT QUESTIONS 7.3

1. Four particles, each of mass **m**, are fixed at the corners of a square whose each side is of length r. Calculate the moment of inertia about an axis passing through one of the corners and perpendicular to the plane of the square. Calculate also the moment of inertia about an axis which is along one of the sides. Verify your result by using the theorem of perpendicular axes.



2. Calculate the radius of gyration of a solid sphere if the axis is a tangent to the sphere. (You may use Table 7.2)

7.4 ANGULAR MOMENTUM

From Table 7.3 you may recall that rotational analogue of linear momentum is angular momentum. To understand its physical significance, we would like you to do an activity.

ACTIVITY 7.4

If you can get hold of a stool which can rotate without much friction, you can perform an interesting experiment. Ask a friend to sit on the stool with her arms folded. Make the stool rotate fast. Measure the speed of rotation. Ask your friend to stretch her arms and measure the speed again. Do you note any change in the speed of rotation of the stool? Ask her to fold her arms once again and observe the change in the speed of the stool.

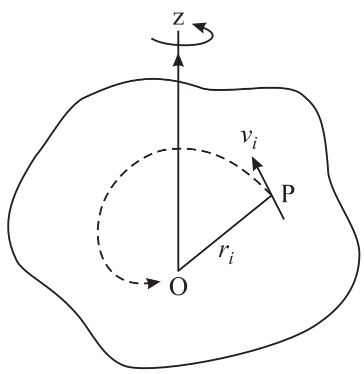


Fig. 7.23 : A rigid body rotating about an axis through 'O'

Let us try to understand why we expect a change in the speed of rotation of the stool in two cases : sitting with folded and stretched hands. For this, let us again consider a rigid body rotating about an axis, say z–axis through a fixed point O in the body. All the points of the body describe circular paths about the axis of rotation with the centres of the paths on the axis and have angular velocity ω. Consider a particle like P at distance r_i from the axis (Fig. 7.20). Its linear velocity is $v_i = r_i\omega$ and its momentum is therefore $m_i r_i \omega$. ***The product of linear momentum and the distance from the axis is called angular momentum, denoted by L.*** If we sum this product for all the particles of the body, we get

$$L = \sum_i m_i \omega\, r_i\, r_i = \left(\sum_i m_i\, r_i^2\right)\omega$$

$$= I\,\omega \qquad (7.23)$$

Remember that the angular velocity is the same for all the particles and the term within brackets is the moment of inertia. Like the linear momentum, the angular momentum is also a vector quantity. Eqn. (7.23) gives only the component of the vector **L** along the axis of rotation. It is important to remember that I must refer to the same axis. The unit of angular momentum is kg m^2 s^{-1}

Notes

Recall now that the rate of change of ω is α and I α = τ. Therefore, ***the rate of change of angular momentum is equal to torque.*** In vector notation, we write the equation of motion of a rotating body as

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = I\frac{d\omega}{dt} = I\,\alpha \qquad (7.24)$$

7.4.1 Conservation of angular momentum

Eqn. (7.24) shows that ***if there is no net torque acting on the body,*** $\frac{d\mathbf{L}}{dt} = 0$. ***This means that there is no change in angular momentum, i.e. the angular momentum is constant. This is the principle of conservation of angular momentum.*** Along with the conservation of energy and linear momentum, this is one of the most important principles of physics.

The principle of conservation of angular momentum allows us to answer questions such as : How the direction of toy umbrella floating in air remains fixed? The trick is to make it rotate and thereby impart it some angular momentum. Once it goes in air, there is no torque acting on it. Its angular momentum is then constant. Since angular momentum is a vector quantity, its constancy implies fixed direction and magnitude. Thus, the direction of the toy umbrella remains fixed while it is in air.

In the case of your friend on the rotating stool; when no net torque acts on the stool, the angular momentum of the stool and the person on it must be conserved. When the arms are stretched, she causes the moment of inertia of the system to increase. Eqn. (7.23) then implies that the angular velocity must decrease. Similarly, when she folds her arms, the moment of inertia of the system decreases. This causes the angular velocity to increase. Note that the change is basically caused by the change in the moment of inertia due to change in distance of particles from the axis of rotation.

Let us look at a few more examples of conservation of angular momentum. Suppose we have a spherical ball of mass M and radius R. The ball is set rotating by applying a torque on it. The torque is then removed. When there is no external torque, whatever angular momentum the ball has acquired must be conserved. Since moment of inertia of the ball is (2/5) $M R^2$ (Table 7.2), its angular momentum is given by

$$L = \frac{2}{5} M R^2 \omega \qquad (7.25)$$

where ω is its angular velocity. Imagine now that the radius of the ball somehow decreases. To conserve its angular momentum, ω must increase and the ball must rotate faster. This is what really happens to some stars, such as those which become pulsars (see Box on page 176).

What would happen if the radius of the ball were to increase suddenly?

You can again use Eqn.(7.25) to show that if R increases, ω must decrease to conserve angular momentum. If instead of radius, the moment of inertia of the system changes some how, ω will change again. For an interesting effect of this kind see Box below

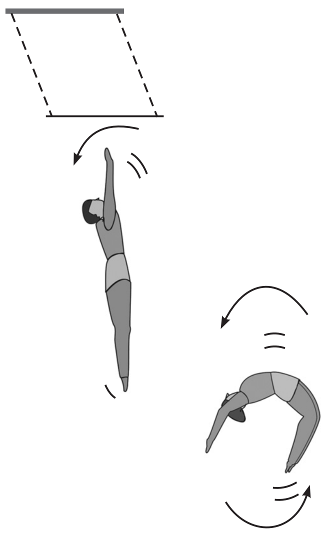


Fig. 7.24 : Diver, Sommer saulting after jumping off the diving boards.

The length of the day is not constant

Scientists have observed very small and irregular variations in the period of rotation of the earth about its axis, i.e. the length of the day. One of the causes that they have identified is weather. Due to changes in weather, large scale movement in the air of the earth's atmosphere takes place. This causes a change in the mass distribution around the axis of the earth, resulting in a change in the moment of inertia of the earth. Since the angular momentum of the earth $L = I\,\omega$ must be conserved, a change in I means a change in rotational speed of the earth, or in the length of the day.

Acrobats, skaters, divers and other sports persons make excellent use of the principle of conservation of angular momentum to show off their feats. You must have seen divers jumping off the diving boards during swimming events in national or international events such as Asian Games, Olympics or National meets. At the time of jumping, the diver gives herself a slight rotation, by which she acquires some angular momentum. When she is in air, there is no torque acting on her and therefore her angular momentum must be conserved. If she folds her body to decrease her moment of inertia (Fig. 7.24) her rotation must become faster. If she unfolds her body, her moment of inertia increases and she must rotate slowly. In this way, by controlling the shape of her body, the diver is able to demonstrate her feat before entering into pool of water.

Example 7.7 : Shiela stands at the centre of a rotating platform that has frictionless bearings. She holds a 2.0 kg object in each hand at 1.0 m from the axis of rotation of the system. The system is initially rotating at 10 rotations per minute. Calculate a) the initial angular velocity in rad s^{-1}, b) the angular velocity after the objects are brought to a distance of 0.2 m from the axis of rotation, and (c) change in the kinetic energy of the system. (d) If the kinetic energy has increased, what is the cause of this increase? (Assume that the moment of inertia of Shiela and platform I_{SP} stays constant at 1.0 kg m^2.)

Solution : (a) 1 rotation = 2π radian

$$\therefore \text{ initial angular velocity } \omega = \frac{10 \times 2\pi \text{ radian}}{60 \text{ s}} = 1.05 \text{ rad s}^{-1}$$

(b) The key idea here is to use the law of conservation of angular momentum. The initial moment of inertia $I_i = I_{SP} + m r_i^2 + m r_i^2$

$$= 1.0 \text{ kg m}^2 + (2.0 \text{ kg}) \times (1 \text{ m}^2) + (2.0 \text{ kg}) \times (1 \text{ m}^2)$$

$$= 5.0 \text{ kg m}^2.$$

After the objects are brought to a distance of 0.2 m, final moment of inertia.

$$I_t = I_{SP} + mr_f^2 + mr_f^2$$

$$= 1.0 \text{ kg m}^2 + 2.0 \text{ kg} \times (0.2)^2 \text{ m}^2 + 2.0 \text{ kg} \times (0.2)^2 \text{ m}^2$$

$$= 1.16 \text{ kg m}^2.$$

Conservation of angular momentum requires that

$$I_i \omega_i = I_f \omega_f$$

or

$$\omega_f = \frac{I_i \omega_i}{I_f}$$

$$= \frac{(5.0 \text{ kg m}^2) \times 1.05 \text{ rad s}^{-1}}{1.16 \text{ kg m}^2}$$

$$= 4.5 \text{ rad s}^{-1}$$

Suppose the change in kinetic energy of rotation is ΔE. Then

$$\Delta E = \frac{1}{2} I_f \omega_f^2 - \frac{1}{2} I_i \omega_i^2$$

$$= \frac{1}{2} \times 1.16 \text{ kg m}^2 \times (4.5)^2 (\text{rad s}^{-1})^2 - \frac{1}{2} \times 5.0 \text{ kg m}^2 \times (1.05)^2 (\text{rad s}^{-1})^2$$

$$= 9.05 \text{J}$$

Since final kinetic energy is higher than the initial kinetic energy, there is an increase in the kinetic energy of the system.

(d) When Shiela pulls the objects towards the axis, she does work on the system. This work goes into the system and increases its kinetic energy.

INTEXT QUESTIONS 7.4

1. A hydrogen molecule consists of two identical atoms, each of mass m and separated by a fixed distance d. The molecule rotates about an axis which is halfway between the two atoms, with angular speed ω. Calculate the angular momentum of the molecule.

2. A uniform circular disc of mass 2.0 kg and radius 20 cm is rotated about one of its diameters at an angular speed of 10 rad s^{-1}. Calculate its angular momentum about the axis of rotation.

3. A wheel is rotating at an angular speed ω about its axis which is kept vertical. Another wheel of the same radius but half the mass, initially at rest, is slipped on the same axle gently. These two wheels then rotate with a common speed. Calculate the common angular speed.

4. It is said that the earth was formed from a contracting gas cloud. Suppose some time in the past, the radius of the earth was 25 times its present radius. What was then its period of rotation on its own axis?

7.5 SIMULTANEOUS ROTATIONAL AND TRANSLATIONAL MOTIONS

We have already noted that if a point in a rigid body is not fixed, it can possess rotational motion as well as translational motion. The general motion of a rigid body consists of both these motions. Imagine the motion of an automobile wheel on a plane horizontal surface. Suppose you are observing the circular face (Fig.7.25). Fix your attention at a point P and at the centre C of the circular face. Remember that the centre of mass of the wheel lies at the centre of its axis and C is the end point of the axis. As it rolls, you would notice that point P rotates round the point C. The point C itself gets translated in the direction of motion. So the wheel has both the rotational and translational motions. If point C or the centre of mass gets translated with velocity υ_{cm}, the kinetic energy of translation is

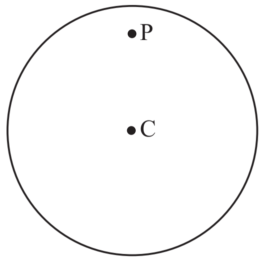


Fig. 7.25

$$(KE)_{tr} = \frac{1}{2} M \upsilon_{cm}^2 \qquad (7.26)$$

where M is the mass. And if ω is the angular speed of rotation, the kinetic energy of rotation is

$$(KE)_{rot} = \frac{1}{2} I \omega^2 \qquad (7.27)$$

where I is the moment of inertia. The total energy of the body due to translation and rotation is the sum of these two kinetic energies. An interesting case, where both translational and rotational motion are involved, is the motion of a body on an inclined plane.

Example 7.8 : Suppose a rigid body has mass M, radius R and moment of inertia I. It is rolling down an inclined plane of height h (Fig.7.26). At the end of its journey, it has acquired a linear speed υ and an angular speed ω. Assume that the loss of energy due to friction is small and can be neglected. Obtain the value of υ in terms of h.

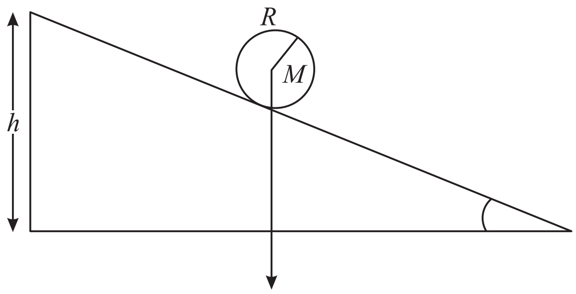


Fig. 7.26 : Motion of a rigid body on an inclined plane

Solution : The principle of conservation of energy implies that the sum of the kinetic energies due to translation and rotation at the foot of the inclined plane must be equal to the potential energy that the body had at the top of the inclined plane. Therefore,

$$\frac{1}{2} M\upsilon^{2} + \tfrac{1}{2} \mathrm{I}\, \omega^2 = M\, g\, h \qquad (7.28)$$

If the motion is pure rolling and there is no slipping, we can write $\upsilon = R\,\omega$. Inserting this expression is Eqn. (7.28), we get

$$\frac{1}{2} M\, \upsilon^2 + \frac{1}{2}\, I\, \frac{\upsilon^2}{R^2} = M\, g\, h \qquad (7.29)$$

To take a simple example, let the body be a hoop. Table 7.2 shows that its moment of inertia about its own axis is MR^2. Eqn.(7.29) then gives

$$\frac{1}{2} M\, \upsilon^2 + \frac{1}{2} \frac{M\, R^2\, \upsilon^2}{R^2} = M\, \mathrm{g}\, h$$

or

$$\upsilon = \sqrt{g\, h} \qquad (7.30)$$

Do you notice any thing interesting in this equation? ***The linear velocity is independent of mass and radius of the hoop. Its means that a hoop of any material and any radius rolls down with the same speed on the inclined plane.***

INTEXT QUESTIONS 7.5

1. A solid sphere rolls down a slope without slipping. What will be its velocity in terms of the height of the slope?

2. A solid cylinder rolls down an inclined plane without slipping. What fraction of its kinetic energy is translational? What is the magnitude of its velocity after falling through a height *h*?

3. A uniform sphere of mass 2 kg and radius 10cm is released from rest on an inclined plane which makes an angle of 30^0 with the horizontal. Deduce its (a) angular acceleration, (b) linear acceleration along the plane, and (c) kinetic energy as it travels 2m along the plane.

Secret of Pulsars

An interesting example of the conservation of angular momentum is provided by pulsating stars. These are called pulsars. These stars send pulses of radiation of great intensity towards us. The pulses are periodic and the periodicity is extremely precise. The time periods range between a few milliseconds to a few seconds. Such short time periods show that the stars are rotating very fast. Most of the matter of these stars is in the form of neutrons. (The neutrons and protons are the building blocks of the atomic nuclei.) These stars are also called neutron stars. These stars represent the last stage in their life. The secret of their fast rotation is their tiny size. The radius of a typical neutron star is only 10 km. Compare this with the radius of the Sun, which is about 7×10^5 km. The Sun rotates on its axis with a period of about 25 days. Imagine that the Sun suddenly shrinks to the size of a neutron star without any change in its mass. In order to conserve its angular momentum, the Sun will have to rotate with a period as short as the fraction of a millisecond.

WHAT YOU HAVE LEARNT

- A rigid body can have rotational as well as translational motion.
- The equation of translational motion far a rigid body may be written in the same form as for a single particle in terms of the motion of its centre of mass.
- If a point in the rigid body is fixed, then it can possess only rotational motion.
- The moment of inertia about an axis of rotation is defined as $\sum_i m_i\ r_i^2$.
- The moment of inertia plays the same role in rotational motion as does the mass in linear motion.

- The turning effect of a force **F** on a rigid body is given by the torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$.
- Two equal and opposite forces constitute a couple. The magnitude of turning effect of torque is equal to the product of one of the forces and the perpendicular distance between the line of action of forces.
- The application of an external torque changes the angular momentum of the body.
- When no net torque acts on a body, the angular momentum of the body remains constant.
- When a cylindrical or a spherical body rolls down an inclined plane without slipping, its speed is independent of its mass and radius.

TERMINAL EXERCISE

1. The weight Mg of a body is shown generally as acting at the centre of mass of the body. Does this mean that the earth does not attract other particles?
2. Is it possible for the centre of mass of a body to lie outside the body? Give two examples to justify your answer?
3. In a molecule of carbon monoxide (CO), the nuclei of the two atoms are 1.13×10^{-10}m apart. Locate of the centre of mass of the molecule.
4. A grinding wheel of mass 5.0 kg and diameter 0.20 m is rotating with an angular speed of 100 rad s^{-1}. Calculate its kinetic energy. Through what distance would it have to be dropped in free fall to acquire this kinetic energy? (Take g = 10.0 m s^{-2}).
5. A wheel of diameter 1.0 m is rotating about a fixed axis with an initial angular speed of 2rev s^{-1}. The angular acceleration is 3 rev s^{-2}.

   (a) Compute the angular velocity after 2 seconds.

   (b) Through what angle would the wheel turned during this time?

   (c) What is the tangential velocity of a point on the rim of the wheel at t = 2 s?

   (d) What is the centripetal acceleration of a point on the rim of the wheel at t = 2 s?
6. A wheel rotating at an angular speed of 20 rads^{-1} is brought to rest by a constant torque in 4.0 seconds. If the moment of inertia of the wheel about

the axis of rotation is 0.20 kg m^2, calculate the work done by the torque in the first two seconds.

7. Two wheels are mounted on the same axle. The moment of inertia of wheel A is 5×10^{-2} kg m^2, and that of wheel B is 0.2 kg m^2. Wheel A is set spinning at 600 rev min^{-1}. while wheel B is stationary. A clutch now acts to join A and B so that they must spin together.
   (a) At what speed will they rotate?
   (b) How does the rotational kinetic energy before joining compare with the kinetic energy after joining?
   (c) What torque does the clutch deliver if A makes 10 revolutions during the operation of the clutch?
8. You are given two identically looking spheres and told that one of them is hollow. Suggest a method to detect the hollow one.
9. The moment of inertia of a wheel is 1000 kg m^2. Its rotation is uniformly accelerated. At some instant of time, its angular speed is 10 rad s^{-1}. After the wheel has rotated through an angle of 100 radians, the angular velocity of the wheel becomes 100 rad s^{-1}. Calculate the torque applied to the wheel and the change in its kinetic energy.
10. A disc of radius 10 cm and mass 1kg is rotating about its own axis. It is accelerated uniformly from rest. During the first second it rotates through 2.5 radians. Find the angle rotated during the next second. What is the magnitude of the torque acting on the disc?

ANSWERS TO INTEXT QUESTIONS

7.1

1. Yes, because the distances between points on the frame cannot change.
2. No. Any disturbance can change the distance between sand particles. So, a heap of sand cannot be considered a rigid body.

7.2

1. The coordinates of given five masses are A (–1, –1), B (–5, –1), C (6, 3), D (2, 6) and E (–3, 0) and their masses are 1 kg, 2kg, 3kg, 4kg and 5kg respectively.

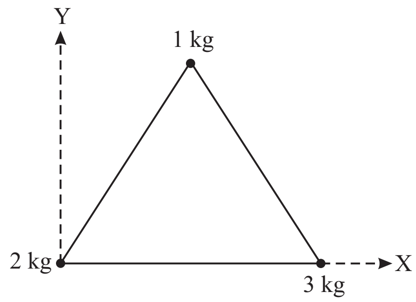
   Hence, coordinates of centre of mass of the system are

$$x = \frac{-1\times1-5\times2+6\times3+2\times4-3\times5}{1+2+3+4+5} = 0$$

$$y = \frac{-1\times1-1\times2+3\times3+4\times6+0\times5}{1+2+3+4+5} = \frac{30}{15} = 2.0$$

2. Let the three particle system be as shown in the figure here. Consider axes to be as shown with 2 kg mass at the origin.

$$x = \frac{2\times0+1\times0.5+3\times1}{1+2+3} = \frac{3.5}{6}\ \text{m} = 0.5\ \text{m}$$



$$y = \frac{2\times0+1\times\frac{\sqrt{3}}{2}+3\times0}{1+2+3} = \frac{\sqrt{3}}{12}\ \text{m}$$

Hence, the co-ordinates of the centre of mass are $\left(\frac{3.5}{6}, \frac{\sqrt{3}}{12}\right)$

3. Let the two particles be along the x-axis and let their x-coordinates be o and x. The coordinate of CM is

$$\text{X} = \frac{m_1\times0+m_2\times x}{m_1+m_2} = \frac{m_2x}{m_1+m_2}, \ \text{Y} = 0$$

X is also the distance of m_1 from the CM. The distance of m_2 from CM is

$$x - \text{X} = x - \frac{m_2x}{m_1+m_2} = \frac{m_1x}{m_1+m_2}$$

$$\therefore \frac{\text{X}}{x+\text{X}} = \frac{m_2}{m_1}$$

Thus, the distances from the CM are inversely proportional to their masses.

7.3

1. Moment of inertia of the system about an axis perpendicular to the plane passing through one of the corners and perpendicular to the plane of the square,

$= m\,r^2 + m\,(2\,r^2) + m\,r^2 = 4\,m\,r^2$

M.I. about the axis along the side $= m\,r^2 + m\,r^2 = 2\,m\,r^2$

Verification : Moment of inertia about the axis QP = $m\,r^2 + m\,r^2 + 2\,m\,r^2$. Now, according to the theorem of perpendicular axes, MI about SP ($2mr^2$) + MI about QP $2\,m\,r^2$ should be equal to MI about the axis through P and perpendicular to the plane of the square ($4\,m\,r^2$). Since it is true, the results are verified.

2. M.I. of solid sphere about an axis tangential to the sphere

$$= \frac{2}{5}\,M\,R^2 + M\,R^2 = \frac{7}{5}\,M\,R^2 \text{ according to the theorem of parallel axes.}$$

If radius fo gyration is K, then $M\,K^2 = \frac{7}{5}\,M\,R^2$. So,

Radius of gyration $K = R\sqrt{\frac{7}{5}}$

7.4

1. Angular momentum

$$L = \left(m\frac{d^2}{4} + m\frac{d^2}{4}\right)\omega$$

$$L = \frac{m\,d^2\omega}{2}$$

2. Angular momentum about an axis of rotation (diameter).

$$L = I\,\omega = m\,\frac{r^2}{4} \times \omega$$

as M.I about a diameter $= \dfrac{m\,r^2}{4}$

$$\therefore\ L = 20\text{ kg} \times \frac{(0.2)^2 m^2}{4} \times 10\text{ rad s}^{-1} = 0.2\text{ kg m}^2\text{ s}^{-1}.$$

3. According to conservation of angular momentum

$$I_1\omega = (I_1 + I_2)\,\omega_1$$

where I_1 is M.I. of the original wheel and I_2 that of the other wheel, ω is the initial angular speed and ω_1 is the common final angular speed.

$$m\,r^2\,\omega = \left(m\,r^2 + \frac{m}{2}r^2\right)\omega_1$$

$$\omega = \frac{3}{2}\,\omega_1 \Rightarrow \omega_1 = \frac{2}{3}\,\omega$$

4. Let the present period of revolution of earth be T and earlier be T_0. According to the conservation of angular momentum.

$$\frac{2}{5}M\,(25\,R)^2 \times \left(\frac{2\pi}{T_0}\right) = \frac{2}{5}\,M\,R^2 \times \left(\frac{2\pi}{T}\right)$$

$$= \frac{2}{5} M R^2 \times \left(\frac{2\pi}{T}\right)$$

It gives, $T_0 = 6.25\ T$

Thus, period of revolution of earth in the past T_0 = 6.25 times the present time period.

7.5

1. Using ($I = \frac{2}{5} M R^2$), Eqn. (7.29) for a solid sphere

$$\frac{1}{2} m\upsilon^2 + \frac{1}{2} I\,\omega^2 = m\,g\,h$$

or, $\frac{1}{2} m\upsilon^2 + \frac{1}{2} \times \frac{2}{5} m\,r^2 \,.\, \frac{\upsilon^2}{r^2} = m\,g\,h$

$\because\ \omega = \upsilon/r.$

It gives $\upsilon = \sqrt{\frac{10}{7}.g\,h}$

2. For a solid cylinder, $I = \frac{m\mathrm{R}^2}{2}$

$\therefore$ Total K.E $\frac{1}{2} m\,\upsilon^2 + \frac{1}{2} I\,\omega^2 = \frac{1}{2} m\,\upsilon^2 + \frac{1}{2}\,\frac{m\mathrm{R}^2}{2} \,.\, \frac{\upsilon^2}{\mathrm{R}^2} = \frac{3}{4} m\,\upsilon^2$

$\because\ \omega = \upsilon/r$

Hence, fraction of translational K.E. = $\dfrac{\frac{1}{2} m\,\upsilon^2}{\frac{3}{4} m\,\upsilon^2} = \frac{2}{3}$

Proceeding as in Q.1 above : $\upsilon = \sqrt{\frac{4}{3} g\,h}$

Answers to Terminal Problems

3. At a distance 0.64 Å from carbon atom.
4. 125 J, 2.5 m
5. (a) 16 π rad s^{-1} (b) 20 π rad (c) 25 m s^{-1} (d) 1280 m s^{-2}
6. 30 J
7. (a) 4 π rad s^{-1} (b) $E_i = 5\ E_f$ (c) 49 π N m
9. T = 5 × 10^4 N m, KE = 5 × 10^6 J
10. 7.5 rad, τ = 5 × 10^{-2} J